Note that the matrix R is used to *store* the roots in its *rows*. The resulting plot is shown in Figure 2.36. The complex conjugate roots are seen to cross into the right half plane for $K = 1.15$. Hence, the system is *stable* for $1 < K < 1.15$ and *unstable* for $K > 1.15$.

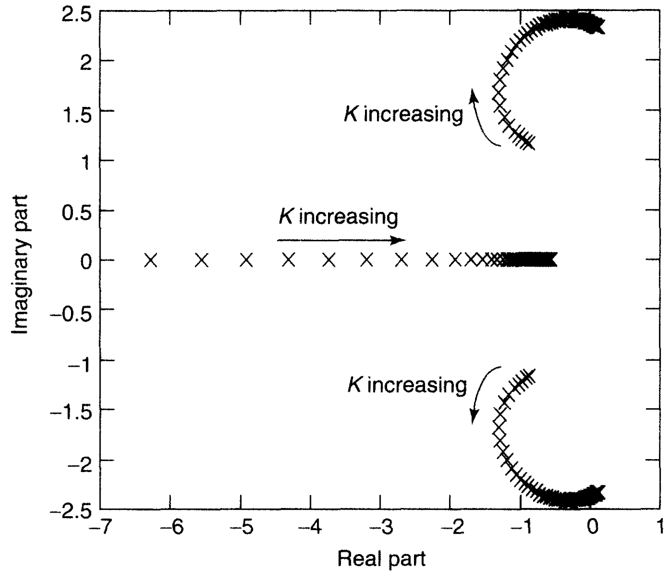

Figure 2.36 Root-locus of the closed-loop system of Example 2.20 as K varies from 0.3 to 1.3

2.10 Nyquist Stability Criterion

The *Nyquist* criterion determines a closed-loop system's stability using the *complex-variable* theory. It employs a polar plot (see Section 2.4) of the *open-loop frequency response*, $G(i\omega)H(i\omega)$, as ω increases from $-\infty$ to ∞. Such a polar plot is called a *Nyquist* plot. In the s-plane, the imaginary axis denotes an increase of ω from $-\infty$ to ∞. Hence, the Nyquist plot of $G(i\omega)H(i\omega)$, as ω increases from $-\infty$ to ∞ is said to be a *mapping* in the $G(s)H(s)$ *plane* (i.e. the plane defined by real and imaginary parts of $G(s)H(s)$) of all the points on the *imaginary axis* in the s-plane. The direction of the Nyquist plot indicates the direction of increasing ω. A polar plot is usually restricted to ω between 0 and ∞. However, note that a polar plot of $G(i\omega)H(i\omega)$ drawn for *negative frequencies* is the *complex conjugate* of the polar plot of $G(i\omega)H(i\omega)$ drawn for *positive frequencies*. In other words, the Nyquist plot is *symmetrical* about the real axis. Hence, the practical technique of plotting Nyquist plot is to first make the polar plot for ω increasing from 0 to ∞, and then draw a *mirror image* of the polar plot (about the real axis) to represent the other half of the Nyquist plot (i.e. the frequency range $-\infty < \omega < 0$). The direction of the *mirror image* polar plot would be clear from the $\omega = 0$ point location on the *original* polar plot, where the two plots should necessarily meet. The two mirror images should also meet at $\omega \to \infty$ and $\omega \to -\infty$, respectively. Hence, $\omega \to \pm\infty$ is a *single* point on the Nyquist plot, and the positive and negative frequency branches of a Nyquist plot form a *closed contour.* It is quite possible that the two polar plots for positive and negative frequencies of some functions may overlap. Figure 2.21 showed the polar plot for positive frequencies. You may verify that a polar plot of the same frequency response for negative frequencies overlaps the curve shown in Figure 2.21, but has an *opposite direction* for increasing ω.

Since the Nyquist plot of $G(s)H(s)$ is a *closed* contour for $s = i\omega$ when $-\infty < \omega < \infty$, whose direction is indicated by *increasing* ω, the only possibility of the positive and negative frequency branches meeting at $\omega \to \pm\infty$ is that the Laplace variable, s, must traverse an *infinite* semi-circle in the *right-half* s-plane. In other words, the region enclosed by the Nyquist plot in the $G(s)H(s)$ plane is a *mapping* of the entire right half s-plane. This fact is depicted in Figure 2.37. The direction of the curves in Figure 2.37 indicate the direction in which the frequency, ω is increasing. The point $G(s)H(s) = -1$ has a special significance in the Nyquist plot, since it denotes the closed-loop characteristic equation, $1 + G(s)H(s) = 0$.

The application of Nyquist stability criteria is restricted to linear, time-invariant control systems with a proper open-loop transfer function, $G(s)H(s)$. Since $G(s)H(s)$ of a linear, time-invariant control system is a *rational function* of s, a point in the s-plane corresponds to *only one* point in the $G(s)H(s)$ plane (such a mapping is called *one-to-one* mapping). Since $G(s)H(s)$ is proper, it implies that $\lim_{s\to\infty} G(s)H(s)$ must be either *zero*, or a non-zero *constant.* A one-to-one mapping from s-plane to the $G(s)H(s)$-plane in which

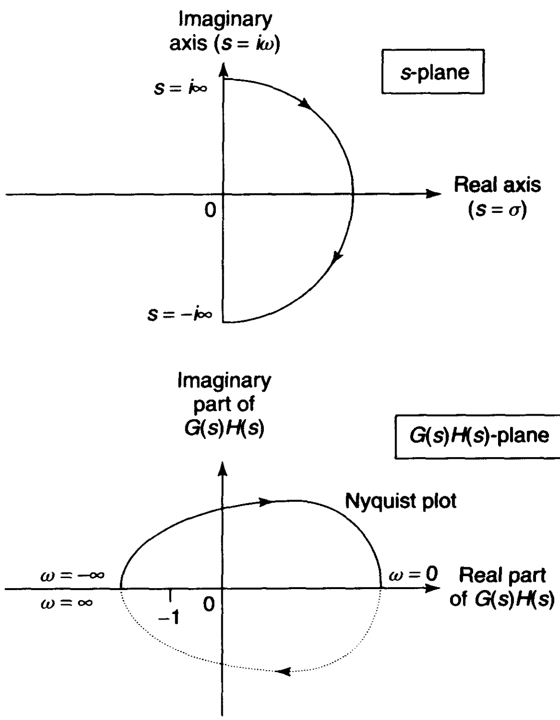

Figure 2.37 The mapping of right-half s-plane into a region enclosed by the Nyquist plot in the $G(s)H(s)$ plane

the limit $\lim_{s\to\infty} G(s)H(s)$ exists and is finite is called a *conformal mapping*. The Nyquist stability criterion is based on a fundamental principle of complex algebra, called *Cauchy's theorem*, which states that if a closed contour with an *anti-clockwise* direction in the s-plane, which *encloses* P poles and Z zeros of $1 + G(s)H(s)$, and which *does not* pass *through* any poles or zeros of $1 + G(s)H(s)$, is *conformally* mapped into a closed contour (i.e. Nyquist plot) in the $G(s)H(s)$ plane, then the latter will encircle the point $G(s)H(s) = -1$ *exactly* N times in an *anti-clockwise* direction, where $N = P - Z$. An anti-clockwise encirclement of $G(s)H(s) = -1$ is considered *positive*, while a clockwise encirclement is considered *negative*. Hence, N could be either *positive*, *negative*, or *zero*. The proof of Cauchy's theorem is beyond the scope of this book, but can be found in D'Azzo and Houpis [2], or in a textbook on complex variables. Applying Cauchy's theorem to the contour enclosing the entire right-half s-plane (shown in Figure 2.37), we find that for closed-loop stability we must have *no zeros* of the closed-loop characteristic polynomial, $1 + G(s)H(s)$, in the right-half s-plane (i.e. $Z = 0$ must hold), and thus $N = P$, which implies that for close-loop stability we must have *exactly* as many *anti-clockwise* encirclements of the point $G(s)H(s) = -1$ as the number poles of $G(s)H(s)$ in the right-half s-plane. (The Nyquist plot shown in Figure 2.37 contains one anti-clockwise encirclement of -1, i.e. $N = 1$. The system shown in Figure 2.37 would be stable if there is exactly one pole of $G(s)H(s)$ in the right-half s-plane.)

Note that Cauchy's theorem does not allow the presence of poles of $G(s)H(s)$ anywhere on the imaginary axis of s-plane. If $G(s)H(s)$ has poles on the imaginary axis, the closed contour in the s-plane (Figure 2.37) should be modified such that it passes *just around* the poles on the imaginary axis. Hence, the closed contour should have loops of infinitesimal radius passing around the poles of $G(s)H(s)$ on the imaginary axis, and the Nyquist plot would be a conformal mapping of such a contour in the $G(s)H(s)$ plane. Studying the effect of each detour around imaginary axis poles on the Nyquist plot is necessary, and could be a tedious process by hand [2].

You can make the Nyquist plot using the polar plot of $G(i\omega)H(i\omega)$ drawn for *positive frequencies* and its complex conjugate for negative frequencies, using the intrinsic MATLAB command *polar* (see Example 2.8). The abscissa of the polar plot must be modified to locate the point $G(s)H(s) = -1$ on the negative real axis (i.e. the radial line corresponding to $\phi = 180^\circ$ in Figure 2.21). (Actually, the polar plot for negative frequencies is not required, since the number of encirclements can be determined from the shape of the polar plot of $G(i\omega)H(i\omega)$ drawn for positive frequencies near the point -1, and the fact that the negative frequency plot is the mirror image of the positive frequency plot.) An easier way of making the Nyquist plot is by the MATLAB Control System Toolbox (CST) command *nyquist(sys)*, where *sys* denotes the LTI object of the open-loop transfer function, $G(s)H(s)$.

Example 2.21

The Nyquist plot of the open-loop transfer function, $G(s)H(s)$ of the system in Figure 2.32 with $G(s) = (2s^2 + 5s + 1)/(s^2 - 2s + 3)$, and $H(s) = 1$ is obtained in Figure 2.38, using the following MATLAB command:

```
>>num=[2 5 1]; den=[1 -2 3]; GH=tf(num,den); nyquist(GH) <enter>
```

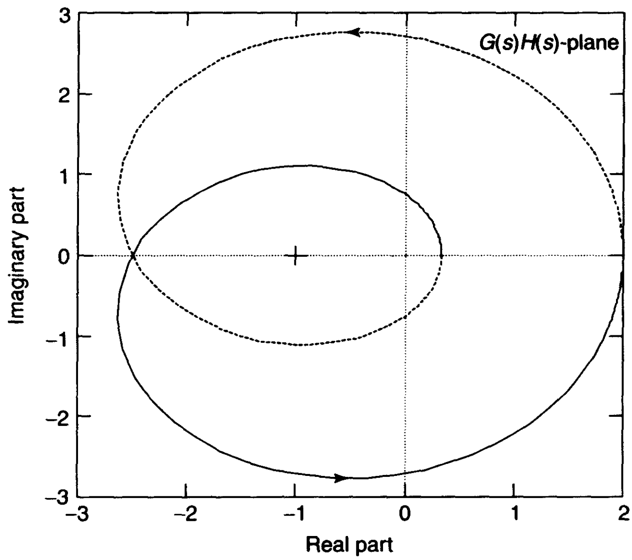

Figure 2.38 Nyquist plot of the closed-loop system in Example 2.21

It is clear from the Nyquist plot in Figure 2.38 that the point -1 is *encircled in the anticlockwise direction exactly twice*. Hence, $N = 2$. The open-loop transfer function $G(s)H(s)$ has *two* poles with positive real parts (i.e. $P = 2$), which is seen by using the MATLAB CST command *damp* as follows:

```
>>damp(den) <enter>

Eigenvalue        Damping   Freq. (rad/sec)
 1.0000+1.4142i   -0.5774    1.7321
 1.0000-1.4142i   -0.5774    1.7321
```

Since $N = P = 2$, Cauchy's theorem dictates that the number of zeros of $1 + G(s)H(s)$ in the right-half plane is $Z = P - N = 0$, which implies that the *closed-loop* transfer function has no poles in the right half plane. Hence by the Nyquist stability criterion, the closed-loop system is *stable*. You can verify that the closed-loop characteristic equation is $1 + G(s)H(s) = 0$, or $3s^2 + 3s + 4 = 0$, resulting in the closed-loop poles $-0.5 \pm 1.0408i$.

A major limitation of the Nyquist stability criterion is that it does not give any indication about the presence of *multiple zeros* of $1 + G(s)H(s)$ at $s = 0$, which cause closed-loop instability due to stability criterion 3 of Section 2.8. Another limitation of the Nyquist stability criterion is that it cannot be applied in cases where the Nyquist plot of $G(s)H(s)$ *passes through* the point $G(s)H(s) = -1$, because in that case the number of encirclements of -1 point are *indeterminate*.