

The gain and phase margins are directly read from Figure 2.43 as shown. However, the crossover frequencies cannot be obtained from the Nichols plot. Apart from giving the gain and phase margins, the Nichols plot can be used to design and analyze the closed-loop frequency response of the system. The intersection of the $G(i\omega)H(i\omega)$ Nichols plot with the closed-loop gain contours give the closed-loop gain frequency response, $|Y(i\omega)/Y_d(i\omega)|$. The frequencies at the intersection points can be obtained from a Bode gain plot of $G(i\omega)H(i\omega)$. However, such a procedure is fairly complicated, and it is easier to get the closed-loop frequency response directly from the Bode plot of $Y(i\omega)/Y_d(i\omega)$.

We have seen how the stability robustness of a closed-loop control system is defined by the gain and phase margins determined from the open-loop transfer function, $G(s)H(s)$. We have considered variations in the overall transfer function, $Y(s)/Y_d(s)$, for defining robustness, and shown that closed-loop systems are more robust to such variations than open-loop systems. Variations in the overall transfer function are called *process noise*. However, closed-loop systems are susceptible to another form of noise, which is absent in open-loop systems, and which arises due to *errors* in *measuring* (and feeding back) the output, $Y(s)$. Such a noise is called *measurement noise*. Invariably, the measurement noise is caused by imperfections in the sensors used to measure the output, and usually occurs at higher frequencies than the natural frequencies of the closed-loop system. To reduce the sensitivity of a closed-loop system to measurement noise (or to make the system robust with respect to measurement noise), the frequency response of the closed-loop system must have *smaller gains at higher frequencies*. This requirement results in the magnitude Bode plot of the closed-loop system *decaying rapidly* (or *rolling-off*) with frequency, at high frequencies. Hence, a controller transfer function must be selected to provide not only good gain and phase margins, which indicate the closed-loop system's robustness to *process noise*, but also a large decay (or *roll-off*) of the gain at high frequencies, indicating robustness due to the measurement noise.

While designing a closed-loop system, one is not only interested in stability in the presence of modeling uncertainties (such as process and measurement noise), but also in maintaining a desired level of *performance*, i.e. one would like to achieve *performance robustness* as well. It can be appreciated that for achieving performance robustness, we should first achieve stability robustness (i.e. there is no point in talking about the performance of an unstable control system). Criteria for achieving stability and performance robustness of *multivariable systems* are more generally expressed with the use of modern *state-space* methods (rather than the classical single-input, single-output, frequency domain procedures), as discussed in Chapter 7.

2.12 Closed-Loop Compensation Techniques for Single-Input, Single-Output Systems

We have seen in the previous sections how the steady-state error, stability, and gain and phase margins can be affected by changing the controller transfer function, $H(s)$, in a

typical single-input, single-output closed-loop system of Figure 2.32. Since we can use $H(s)$ to *compensate* for the poor characteristics of a plant, $G(s)$, such a controller, $H(s)$, is called a *compensator*, and the procedure of selecting a controller, $H(s)$, in order to remove the *deficiencies* of the plant, $G(s)$, (such as improving stability, performance, and robustness), is called *closed-loop compensation*. *Classical* control techniques of designing single-input, single-output control systems based upon their frequency response characteristics largely rely upon *closed-loop compensation*. We will consider some commonly employed closed-loop compensation techniques. Closed-loop compensation is generally of two types: cascade (or series) compensation in which $H(s)$ is placed in series with the plant (Figure 2.32), and feedback (or parallel) compensation in which $H(s)$ is placed in the feedback path of $G(s)$ (Figure 2.35). Cascade and feedback compensation represent two alternatives for achieving the same closed-loop characteristics. Where we insert the compensator in the control system depends largely upon the physical aspects of implementation. We study only cascade compensation techniques here, which can be easily extended to feedback compensation.

2.12.1 Proportional-integral-derivative compensation

A popularly used compensator with the transfer function, $H(s) = (K_D s^2 + K_P s + K_I)/s$, where K_D , K_P , and K_I are constants, is called the *proportional plus integral plus derivative* (PID) *compensator*, because its transfer function can be expressed as $H(s) = K_P + K_D s + K_I/s$, signifying that the output of the controller, $U(s)$, is a sum of its input, $E(s)$, multiplied by *constant* K_P , the *integral* of the input times K_I , and the *derivative* of the input times K_D , i.e. $U(s) = K_P E(s) + K_I E(s)/s + K_D s E(s)$ (recall that $E(s)/s$ is the Laplace transform of the *integral* of the function, $e(t)$, and $sE(s)$ is the Laplace transform of the derivative of $e(t)$ if the initial conditions are zero). PID compensators are very common in industrial applications due to their good *robustness* over a wide frequency range. The presence of the integral term, K_I/s , in $H(s)$ increases the type of the closed-loop system due to the pole at the origin, thereby reducing the steady-state error. The derivative term, $K_D s$, and the proportional term, K_P , can be used to place two *zeros* of $H(s)$ at suitable locations, to change the phase characteristics of the closed-loop system. Let us consider an example of PID compensation.

Example 2.23

Consider a hard-disk read/write head positioning system with the following transfer function:

$$Y(s)/U(s) = G(s) = 700/(s^2 + 15s + 100\,000) \quad (2.165)$$

The output, $Y(s)$, is the angular position of the head in radians, while the input, $U(s)$, is the current in milli-Amperes (mA) supplied to the head positioning solenoid. The poles, natural frequencies, and damping-ratios for this second order plant are as follows:

```
>>num=700; den = [1 15 1e5]; damp(den) <enter>
```

Eigenvalue	Damping	Freq. (rad/sec)
-7.5000e+000+3.1614e+002i	2.3717e-002	3.1623e+002
-7.5000e+000-3.1614e+002i	2.3717e-002	3.1623e+002

Note that the damping is very small, and hence the head positioning system, will oscillate a lot before coming to a steady-state. Since the plant is of type 0, the steady-state error due to a step input will be non-zero. To make the steady-state error zero for a step input, we initially choose a PID compensator with $K_D = 0$, $K_P = 1$, and $K_I = 1100$, which makes the controller transfer function the following:

$$H(s) = (s + 1100)/s \quad (2.166)$$

Since $H(s)$ has no derivative term, it is essentially a *proportional-plus-integral* (PI) compensator, with a zero at $s = -1100$ and a pole at $s = 0$. Connecting $H(s)$ and $G(s)$ in series, and then closing the feedback loop as shown in Figure 2.32, we get the following closed-loop transfer function:

$$Y(s)/Y_d(s) = (700s + 770\,000)/(s^3 + 15s^2 + 100\,700s + 770\,000) \quad (2.167)$$

The Bode plots of the plant, $G(s)$, the compensator, $H(s)$, and the closed-loop system are obtained as follows, and are shown in Figure 2.44:

```
>>G=tf(num,den); w=logspace(-1,4); [mag1,phase1,w]=bode(G,w); %  
Bode plot of the plant <enter>
```

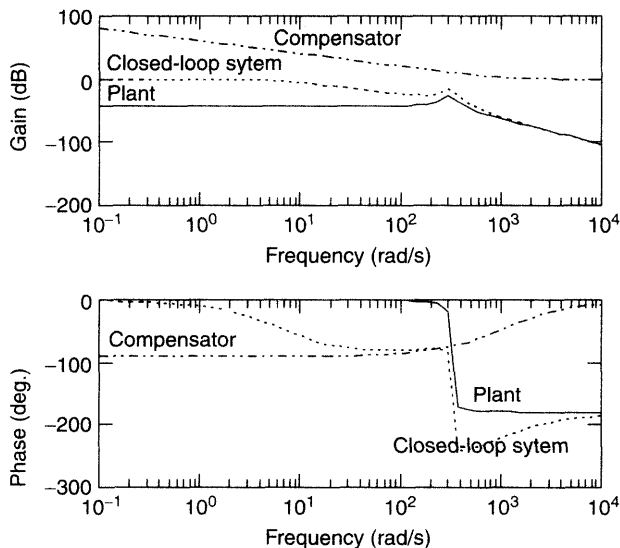


Figure 2.44 Bode plots of the plant, PI compensator, and closed-loop system for the hard-disk read/write head

```
>>n1=[1 1100]; d1=[1 0];H=tf(n1,d1); [mag2,phase2,w]=bode(H,w); %
  Bode plot of the compensator <enter>

>>nCL=[700 770 000]; dCL=[1 15 100 700 770 000]; GCL=tf(nCL,dCL);
  <enter>

>>[mag3,phase3,w]=bode(GCL,w); % Bode plot of the closed-loop system
  <enter>
```

Note from Figure 2.44 that the PI compensator *increases the gain*, while *decreasing the phase* at low frequencies, and leaves the *high frequency gain* unaffected. The *DC gain* of the closed-loop system is brought to 0 dB, indicating that the step response will have a *zero steady-state error*. There is a *reduction* in the *gain margin* of the closed-loop system due to the increased gain at low frequencies, when compared to that of the plant, indicating a loss of robustness. The plant's natural frequency of 316.23 rad/s is visible as a peak in the gain plot. The closed-loop poles are the following:

```
>>damp(dCL) <enter>
```

Eigenvalue	Damping	Freq. (rad/sec)
-3.6746e+000+3.1722e+002i	1.1583e-002	3.1724e+002
-3.6746e+000-3.1722e+002i	1.1583e-002	3.1724e+002
-7.6507e+000	1.0000e+000	7.6507e+000

Note that the closed-loop damping near plant's natural frequency is *slightly reduced*, while another pole is placed at $s = -7.6507$. How does this closed-loop pole configuration affect the step response of the system? This question is best answered by comparing the plant's step response with that of the closed-loop system. The step responses are calculated using the M-file *stepresp.m* of Table 2.3 as follows:

```
>>[s1,t] = stepresp(num,den,0,0.01,1); % step response of the plant
  <enter>

>>[s2,t] = stepresp(nCL,dCL,0,0.01,1); % step response of the
  closed-loop system <enter>
```

The two step responses are compared in Figure 2.45. Note that while the closed-loop steady-state error is brought to zero, the settling time and the large number of oscillations of the plant are unaffected.

Furthermore, let us compare the gain and phase margins of the closed-loop system with those of the plant, to see how the stability robustness is affected. The gain and phase margins of the plant, with the respective crossover frequencies, are calculated using the command *margin* as follows:

```
>>[gm,pm,wg,wp]=margin(G) <enter>

gm =
  Inf
```

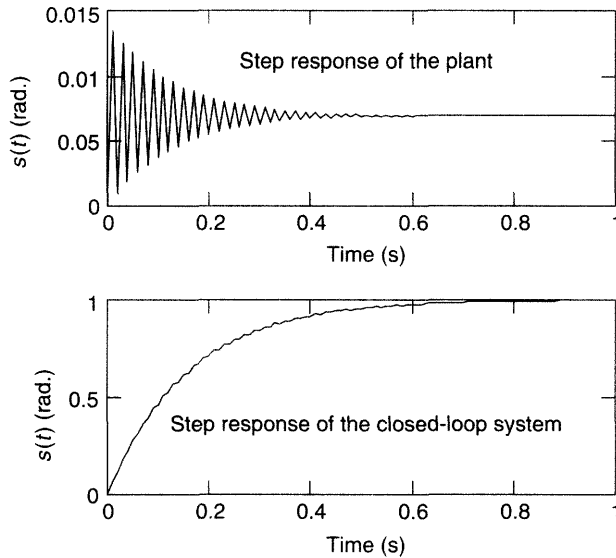


Figure 2.45 Step responses of the plant and the PI compensated closed-loop system for the hard-disk read/write head

```
pm =
    Inf
wg =
    NaN
wp =
    NaN
```

where *inf* denotes ∞ , and *NaN* stands for ‘not a number’, i.e. an *undefined* quantity. The plant thus has *infinite* gain and phase margins, and the corresponding crossover frequencies are thus undefined. The margins for the closed-loop system are the following:

```
>>[gmCL,pmCL,wgCL,wpCL] = margin(GCL) <enter>

gmCL =
    0.9750
pmCL =
   -3.8267
wgCL =
   318.4062
wpCL =
   318.6843
```

Both gain and phase margins of the closed-loop system have been drastically reduced to only 0.975 (−0.22 dB) and -3.8267° , respectively, with the gain and phase crossover frequencies quite close together at 318.7 rad/s and 318.4 rad/s, respectively. Clearly, the closed-loop system is quite less robust than the plant. In summary,

the PI compensator given by Eq. (2.166) not only results in a highly oscillatory response, but also a significant loss in robustness.

Let us now improve the PID compensation by *selecting* the values of K_D , K_P , and K_I , that lead to a *desired* closed-loop response. It is desirable to have a well damped closed-loop response, to reduce the number of oscillations, as well as the settling time. Hence, let us select a damping-ratio of $\zeta = 0.707$ for the closed-loop system, without changing the plant's natural frequency, $\omega_n = 316.23$ rad/s. The general transfer function of the forward path, $G(s)H(s)$, can then be written as follows:

$$G(s)H(s) = 700(K_D s^2 + K_P s + K_I) / [s(s^2 + 15s + 100\,000)] \quad (2.168)$$

Note that the closed-loop system's type is increased from 0 to 1, due to the presence of the pole at $s = 0$ in $G(s)H(s)$. Hence, the steady-state error to step input will be zero, irrespective of the values of K_D , K_P , and K_I . The closed-loop transfer function is $G(s)H(s) / [1 + G(s)H(s)]$, which can be expressed as $N(s)/D(s)$, where $N(s)$ and $D(s)$ are the following numerator and denominator polynomials, respectively:

$$N(s) = 700(K_D s^2 + K_P s + K_I); D(s) = s^3 + (15 + 700K_D)s^2 + (100\,000 + 700K_P)s + 700K_I \quad (2.169)$$

Note that the closed-loop system is of third order. We can write $D(s)$ as a quadratic factor in s multiplied by a first order polynomial in s , as follows:

$$D(s) = (s^2 + 2\zeta\omega_n s + \omega_n^2)(s + p) \quad (2.170)$$

Note from Eq. (2.116) that the step (or impulse) response of a system is *dominated* by the poles with the *smallest real part magnitudes*. If we select the *two* poles of the closed-loop system resulting from the roots of $(s^2 + 2\zeta\omega_n s + \omega_n^2)$ to be *closer* to the imaginary axis than the *third pole* $s = -p$, the closed-loop response would be *dominated* by the quadratic factor in $D(s)$, with $(s + p)$ influencing the closed-loop response by a lesser extent. In such a case, the roots of $(s^2 + 2\zeta\omega_n s + \omega_n^2)$ are called the *dominant poles* of the closed-loop system. Since we have already selected the closed-loop damping-ratio as $\zeta = 0.707$, and natural frequency as, $\omega_n = 316.23$ rad/s, we can choose the pole $s = -p$ to be *further away* from the imaginary axis in the left-half plane by having $p > \zeta\omega_n$ (i.e. $p > 223.57$). Let us take $p = 300$. Substituting the values of ζ , ω_n , and p into Eq. (2.170) and comparing with $D(s)$ given in Eq. (2.169), we get the following values of the PID compensator constants that would result in the desired closed-loop dynamics: $K_I = 42857.14$, $K_D = 1.0459$, $K_P = 191.64$, and the compensator's transfer function is given by

$$H(s) = (1.0459s^2 + 191.64s + 42\,857.14)/s \quad (2.171)$$

With this PID compensator, we get the following closed-loop transfer function:

$$Y(s)/Y_d(s) = (732.15s^2 + 134\,140s + 3 \times 10^7) / [(s + 300)(s^2 + 447.15s + 1\,00\,000)] \quad (2.172)$$

Let us check the closed-loop step response as follows:

```
>>nCL = [732.15 134140 3e7]; dCL = conv([1 300],[1 447.15 1e5]); <enter>
>>[s,t] = stepresp(nCL,dCL,0,0.0005,0.03); plot(t,s) <enter>
```

The resulting closed-loop step response is plotted in Figure 2.46. Note the settling time of 0.025 seconds (as compared to 0.7 seconds for the plant), and a *complete lack* of high frequency oscillations in the step response, with a zero steady-state error. The performance is thus greatly improved by the PID compensation.

Finally, let us determine the gain and phase margins of the PID compensated closed-loop system as follows:

```
>>[gmCL,pmCL,wgCL,wpCL]=margin(GCL) <enter>
```

```
gmCL =
    Inf
pmCL =
    144.8891
wgCL =
    NaN
wpCL =
    583.6209
```

Note that the closed-loop gain margin is infinite, and the phase margin is 144.9° occurring at gain crossover frequency of 583.6 rad/s. Although the phase margin is no longer infinite as for the plant, it is quite large and adequate. The large

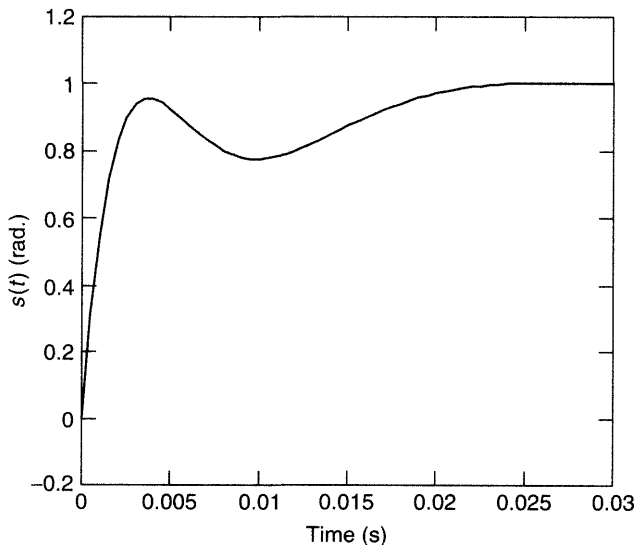


Figure 2.46 Closed-loop step response of the PID compensated hard-disk read/write head positioning system

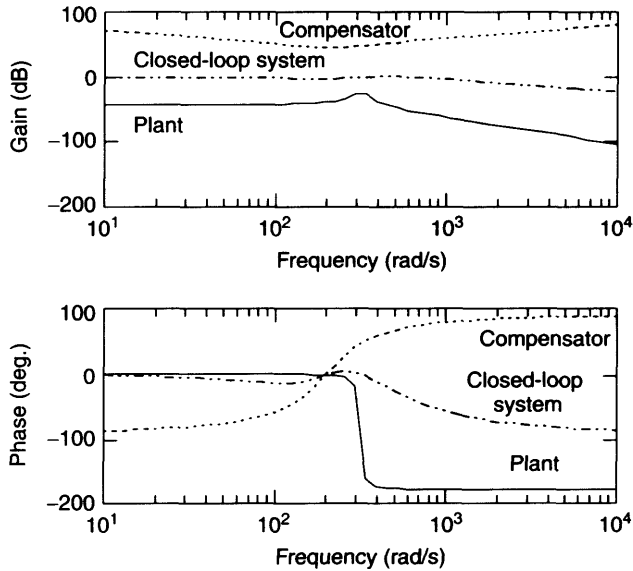


Figure 2.47 Bode plots of the plant, PID compensator, and closed-loop system for the hard-disk read/write head

value of the gain crossover frequency indicates a very fast response, which is evident in Figure 2.46. In short, the PID compensated closed-loop system has a good combination of performance and stability robustness. The Bode plots of the PID compensator and the closed-loop system are compared with those of the plant in Figure 2.47. Note that the PID compensator has a *decreasing* gain at low frequencies, and an *increasing* gain at high frequencies. Also, the PID compensator provides a *phase-lag* at frequencies below the plant's natural frequency, and a *phase-lead* at higher frequencies. The resulting closed-loop gain and phase plots are much *flatter* (compared to the plant). However, due to an increased closed-loop gain at high frequencies, there is an *increased* sensitivity (and decreased robustness) with respect to the high frequency *measurement noise*, which is undesirable.

The process of finding suitable PID constants K_D , K_P , and K_I (as illustrated very simply in Example 2.23) is called *PID tuning*. Often, PID tuning in trying to achieve desirable closed-loop characteristics is an iterative procedure. A PID (or PI) compensator contains an *integrator* (i.e. a pole $s = 0$) which requires special implementation techniques. Mechanically, a pure integration is possible using a *rate-integrating gyroscope* – a commonly used (but expensive) device in aircraft, missile, and spacecraft control systems. Figure 2.48(a) depicts the schematic diagram of a *single degree of freedom* rate gyroscope, with *gimbal angle* (i.e. rotation of the wheel assembly about the x -axis), θ , as the output and the *angular velocity of the case* about the z -axis, Ω , as the input. If we neglect the stiffness and inertia of the gimbal and wheel about the x -axis, we can write the following transfer

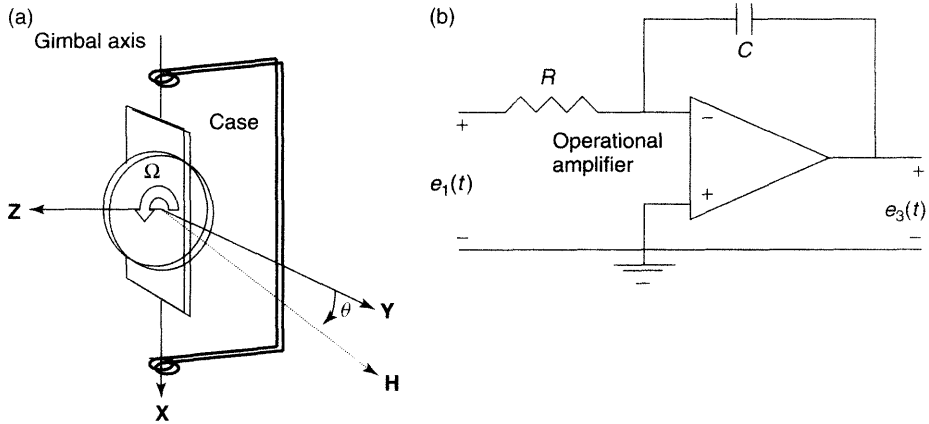


Figure 2.48 (a) Schematic diagram of a single degree of freedom, rate integrating gyroscope with output, $\theta(s)$ and input, $\Omega(s)$. $\mathbf{H}$ is the angular momentum vector with a constant magnitude, H ; (b) Active circuit of an operational amplifier connected as an integrator with input, $e_1(t)$ and output, $e_3(t)$

function for the rate-integrating gyroscope:

$$\theta(s)/\Omega(s) = H/(cs) \quad (2.173)$$

where H is the constant magnitude of the *angular momentum vector*, $\mathbf{H}$, of the wheel about the y -axis, and c is the *viscous damping coefficient* of the gimbal (i.e. rotary mechanism about the x -axis) (see Figure 2.48(a)). Equation (2.173) indicates that the gimbal angle output is the time integral of the input angular velocity of the case.

Modern control systems are relatively inexpensively implemented using either *passive* (unpowered) or *active* (powered) electronic circuits. Figure 2.48(b) shows an active circuit which uses an *operational amplifier* to represent a pure integrator. An operational amplifier has two input voltages, $e_1(t)$ and $e_2(t)$, and an output voltage, $e_3(t)$. In the circuit shown in Figure 2.48(b), $e_2(t) = 0$, and the equation governing the circuit is the following:

$$e_3(t) = -1/(RC) \int e_1(t) dt \quad (2.174)$$

which implies that the output voltage, $e_3(t)$, is the *time integral* of the input voltage, $e_1(t)$, multiplied by the constant $-1/(RC)$. Operational amplifiers can also be used to represent summing junctions, and other useful devices.

Control systems based upon operational amplifiers (and other active circuits), are generally more expensive and sensitive to noise than those based on passive circuits. A simpler alternative to operational amplifier is the *approximation* of the pure integrator by a *passive* circuit, called *lag circuit*. The chief difficulty in implementing PID compensators is the *ideal differentiator* (i.e. the term $K_d s$ in the expression for $H(s)$). An ideal differentiator is difficult to set up, and leads to the amplification of any noise present in the input signal. The noise amplification may interfere with the working of the entire control system. For these reasons, a pure differentiator is never used practically, but only an *approximate* differentiation is implemented using a *passive* circuit (called a *lead circuit*). Hence, the

ideal PID compensator is practically implemented by a combination of lag and lead circuits (called a *lead-lag* compensator), which represent *approximate* integration and differentiation.

2.12.2 Lag, lead, and lead-lag compensation

Consider a compensator with the following transfer function:

$$H(s) = (s + \omega_o)/(s + \omega_o/\alpha) \quad (2.175)$$

where α is a real constant and ω_o is a constant frequency. If $\alpha > 1$, the compensator is called a *lag compensator*, because it always has a *negative* phase angle, i.e. a *phase-lag*. In the limit $\alpha \rightarrow \infty$, the lag compensator approaches an ideal PI compensator. Hence, a lag compensator can be used to approximate a PI compensator in practical implementations. A *lag compensator* is useful in reducing the steady-state error of type 0 plants, and decreasing the gain of the closed-loop system at high frequencies (which is desirable for reducing the sensitivity to the measurement noise). However, lag compensation *slows down* the closed-loop transient response (i.e. increases the settling time). Lag compensation is relatively simple to use, because the *passive circuit* through which it can be implemented is quite inexpensive.

In Eq. (2.175), if $\alpha < 1$, the resulting compensator is called a *lead compensator*, because it always has a *positive* phase angle, i.e. a *phase-lead*. A *lead compensator* is useful for increasing the *speed* of the closed-loop response (i.e. decreasing the *settling time*), and increasing the *phase margin* of the closed-loop system, which also results in smaller overshoots in the transient response. Lead compensation usually requires amplification of error signals, which results in an expensive electrical circuit for implementation. Also, lead compensation increases the gain at high frequencies, which is undesirable due to increased sensitivity to measurement noise. A lead compensator given by the transfer function of Eq. (2.175) would *decrease* the DC gain of the type 0 open-loop transfer function, $G(s)H(s)$, which is undesirable as it would *increase* the steady-state error due to a step input. So that the DC gain of the type 0 open-loop transfer function, $G(s)H(s)$, is *unchanged* with a lead compensator, the lead compensator transfer function is usually multiplied by the factor $1/\alpha$, resulting in

$$H(s) = (s + \omega_o)/(\alpha s + \omega_o) \quad (2.176)$$

Conversely, if we do not wish to *increase* the DC gain of $G(s)H(s)$ (which may be infinite due to poles of $G(s)$ at $s = 0$) with *lag compensation*, we would choose a lag compensator with the transfer function given by Eq. (2.176). Phase lag (or lead) compensation is traditionally employed in a variety of control applications.

To combine the desirable properties of the lead and lag compensators, sometimes it is better to use a compensator which has *both* a phase-lag at low frequencies, *and* a phase-lead at high frequencies. Such a compensator is called a *lead-lag compensator*, and has the transfer function

$$H(s) = [(s + \omega_1)/(s + \omega_1/\alpha)][(s + \omega_2)/(s/\alpha + \omega_2)] \quad (2.177)$$

where $\alpha > 1$, and ω_1 and ω_2 are constant frequencies. Note that the transfer function suggests that the lead-lag compensator consists of a lag compensator in series with a lead compensator. Also, note that the lag part of the transfer function is designed to increase the DC gain of open-loop system, $G(s)H(s)$. The frequencies, ω_1 and ω_2 , and the constant, α , must be selected to achieve desired closed-loop characteristics. It is easy to see that in the limit $\alpha \rightarrow \infty$, the lead-lag compensator approaches an ideal PID compensator.

Figure 2.49 shows the passive circuits used to implement the lag, lead, and lead-lag compensators. In each circuit, $e_1(t)$ and $e_2(t)$ are the input and output voltages, respectively. The transfer function of the *lag circuit* (Figure 2.49(a)) is expressed as

$$H(s) = E_2(s)/E_1(s) = (1 + R_2Cs)/[1 + (R_1 + R_2)Cs] \quad (2.178)$$

Comparing Eqs. (2.178) and (2.176), it is clear that $\omega_o = 1/(R_2C)$ and $\alpha = (R_1 + R_2)/R_2$. The *lead circuit* in Figure 2.49(b) has the transfer function

$$H(s) = E_2(s)/E_1(s) = R_2(1 + R_1Cs)/[(R_1 + R_2) + R_1R_2Cs] \quad (2.179)$$

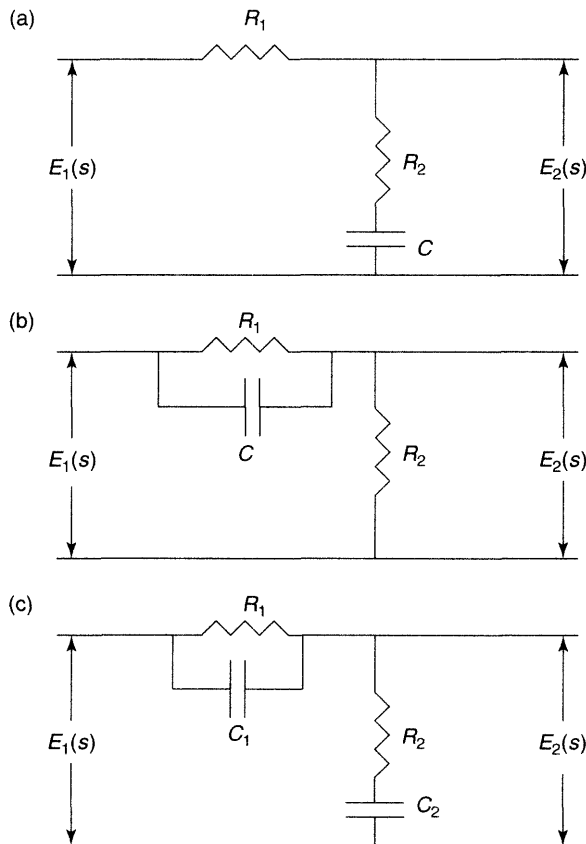


Figure 2.49 (a) Lag circuit with input voltage $E_1(s)$ and output voltage $E_2(s)$; (b) Lead circuit with input voltage $E_1(s)$ and output voltage $E_2(s)$; (c) Lead-lag circuit with input voltage $E_1(s)$ and output voltage $E_2(s)$

which implies that $\omega_o = 1/(R_1 C)$ and $\alpha = R_2/(R_1 + R_2)$. Finally, the transfer function of the *lead-lag* circuit shown in Figure 2.49(c) is the following:

$$H(s) = E_2(s)/E_1(s) = [(1 + R_1 C_1 s)(1 + R_2 C_2 s)] / [(1 + R_1 C_1 s)(1 + R_2 C_2 s) + R_1 C_2 s] \quad (2.180)$$

Comparing Eqs. (2.180) and (2.177), you may verify that $\omega_1 = 1/(R_1 C_1)$, $\omega_2 = 1/(R_2 C_2)$, and α is obtained by factoring the denominator (i.e. solving a quadratic equation) in Eq. (2.180).

Example 2.24

Consider the *roll dynamics* of a fighter aircraft with the following transfer function:

$$\phi(s)/\delta(s) = G(s) = 1000/[s(s + 5)] \quad (2.181)$$

The output, $Y(s) = \phi(s)$, is the *roll-angle*, while the input, $U(s) = \delta(s)$, is the *aileron deflection angle*. The maneuverability of an aircraft depends upon the time taken to achieve a *desired roll-angle*, (or *bank-angle*) $Y_d(s) = \phi_d(s)$. It is required to design a closed-loop control system to achieve a *unit step* desired roll-angle in about three seconds, with a maximum overshoot less than 2 percent. Since the plant has a pole at $s = 0$, it is not *asymptotically stable* and the steady-state condition would *not* be achieved by plant alone, thereby requiring a feedback compensation that would provide asymptotic stability. Since the plant is of type 1 due to the pole at the origin, a simple feedback system with $H(s) = K$ (where K is a constant) will achieve a zero steady-state error to a step desired output. With such a compensator, the closed-loop characteristic equation would be $s^2 + 5s + 1000K = 0$, and the closed-loop poles would be $s_{1,2} = -2.5 \pm (1000K^2 - 6.25)^{1/2}i$, with a damping-ratio, $\zeta = 0.0791/\sqrt{K}$, and natural frequency, $\omega_n = 31.6228\sqrt{K}$. If we want to achieve a settling time, T_s , of 3 seconds, then from Eq. (2.125) we have $T_s = 4/(\zeta\omega_n)$, or $\zeta\omega_n = 4/3$. The closed-loop system with $H(s) = K$ can give us $\zeta\omega_n = 0.0791 \times 31.6228 = 2.5$, or $T_s = 4/2.5 = 1.6$ seconds, which is a *smaller* settling time than required. The maximum overshoot requirement of 2 percent of steady-state response, according to Eq. (2.127) results in the condition $M_p = 1.02 = 1 + \exp[-\zeta\pi/(1 - \zeta^2)^{1/2}]$, or $\zeta = 0.7797$, which requires $\sqrt{K} = 0.0791/0.7797 = 0.1014$, or $K = 0.0103$. Hence, we should have $K \leq 0.0103$ in order to satisfy *both* settling time and maximum overshoot requirements. Let us choose $K = 0.01$, which gives us a closed-loop settling time, $T_s = 1.6$ seconds, and maximum overshoot, $M_p = 1.0172$, or the maximum percentage overshoot of 1.72 percent. Thus, the performance requirements are met quite successfully with $H(s) = 0.01$.

Let us now consider a *lag-compensator* for this task. Since $G(s)$ is of type 1, we do not need to increase the DC gain of $G(s)H(s)$ to reduce the steady-state error to a step input. Hence, we would like to use a lag compensator transfer function of the form $H(s) = (s + \omega_o)/(\alpha s + \omega_o)$, with $\alpha > 1$, which leaves the DC gain of $G(s)H(s)$ unchanged. A candidate lag-compensator transfer function which achieves the desired performance is $H(s) = (s + 0.001)/(100s + 0.001)$, i.e. $\omega_o = 0.001$ rad/s and $\alpha = 100$. The closed-loop transfer function is thus the

following:

$$\phi(s)/\phi_d(s) = 1000(s + 0.001)/(100s^3 + 500s^2 + 1000s + 1) \quad (2.182)$$

The natural frequencies and damping-ratios of the closed-loop system are the following:

```
>>nCL=[0 0 1000 1]; dCL=[100 500 1000 1]; damp(dCL) <enter>
```

Eigenvalue	Damping	Freq. (rad/sec)
-0.0010	1.0000	0.0010
-2.4995+1.9359i	0.7906	3.1615
-2.4995-1.9359i	0.7906	3.1615

Note that there is a closed-loop pole at $s = -0.001$. Since there is a closed-loop zero at the same location, after *canceled* the pole with the zero at $s = -0.001$, the closed-loop transfer function can be written as:

$$\phi(s)/\phi_d(s) = 10/(s^2 + 5s + 10) \quad (2.183)$$

which is the *same* transfer function as that obtained with $H(s) = 0.01$! This is an interesting result, and shows that the lag compensator has resulted in a *third order* closed-loop system, which essentially behaves as a *second order* system due to a pole-zero cancelation. The performance objectives are met by such a lag compensator, as shown above for $H(s) = 0.01$.

Figure 2.50 compares the Bode plots of the plant, the lag compensator, and the closed-loop system. The lag compensator provides a negative phase and a reduction

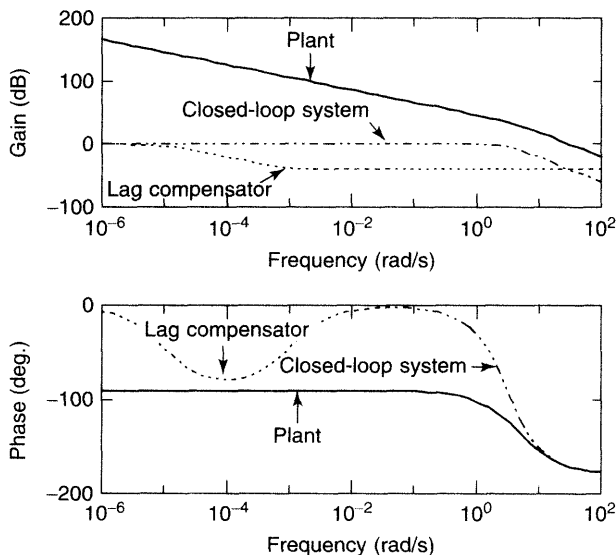


Figure 2.50 Bode plots of the plant, lag compensator, and closed-loop system for the roll control of a fighter aircraft

in gain in the frequencies below $\alpha\omega_0$ (i.e. 0.1 rad/s). Since the closed-loop gain never crosses the 0 dB line, and the closed-loop phase tends asymptotically to -180° , the gain and phase margins are *infinite* for the closed-loop system. Also, note that there is no change in the *slope* of gain reduction with frequency (i.e. *roll-off*) at high frequencies of the closed-loop system, when compared to that of the plant, which implies that there is no reduction in the *robustness* with respect to high-frequency *measurement noise*.

Example 2.25

Consider a large chemical plant with the following transfer function:

$$Y(s)/U(s) = G(s) = 0.025/[(s^2 + 0.3s + 0.01)(s + 0.33)] \quad (2.184)$$

where the output, $Y(s)$, is the temperature, and the input, $U(s)$, is the mass flow-rate of Xylene gas. The natural frequencies of the plant are calculated as follows:

```
>>num=0.025; den=conv([1 0.33],[1 0.3 0.01]); damp(den) <enter>
```

Eigenvalue	Damping	Freq. (rad/sec)
-0.0382	1.0000	0.0382
-0.2618	1.0000	0.2618
-0.3300	1.0000	0.3300

The plant's response is dominated by the pole, $s = -0.0382$, which is very much closer to the imaginary axis than the other two poles. Hence, the plant has a settling time of approximately $T_s = 4/0.0382 = 105$ seconds. The steady-state error of the plant is $e(\infty) = 1 - \lim_{s \rightarrow 0} sY(s) = -6.5758$ for a unit step input. It is required to design a closed-loop control system such that the steady-state error is brought down to less than ± 0.15 , with a maximum overshoot of 10 percent, and a settling time less than 20 seconds to a unit step desired output, $Y_d(s)$. Consider a *lead compensator* with $\omega_0 = 0.15$ rad/s and $\alpha = 0.01$, which results in a compensator transfer function, $H(s) = (s + 0.15)/(0.01s + 0.15)$. The closed-loop transfer function, $G(s)H(s)/[1 + G(s)H(s)]$, is thus

$$Y(s)/Y_d(s) = 0.025(s + 0.15)/(0.01s^4 + 0.1563s^3 + 0.09559s^2 + 0.041383s + 0.004245) \quad (2.185)$$

The closed-loop poles and natural frequencies are the following:

```
>>nCL=0.025*[1 0.15]; dCL=[0.01 0.1563 0.09559 0.041383 0.004245]; damp(dCL) <enter>
```

Eigenvalue	Damping	Freq. (rad/sec)
-0.1358	1.0000	0.1358
-0.2414+0.3873i	0.5289	0.4563
-0.2414-0.3873i	0.5289	0.4563
-15.0115	1.0000	15.0115

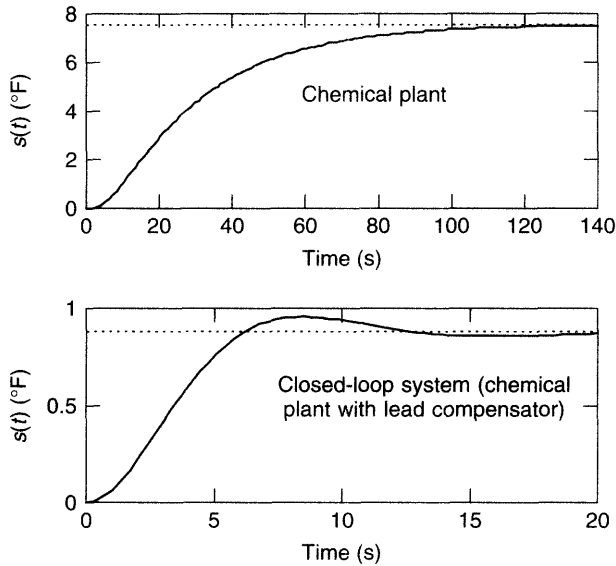


Figure 2.51 Step responses of the chemical plant and the closed-loop system with the lead compensator

Since there is no clearly dominant pole (or conjugate pair of poles), the settling time of the closed-loop system is determined by the first three poles, which indicates a dominant third order system. The step response of the closed-loop system is compared with that of the plant in Figure 2.51, obtained using the M-file *stepresp.m*. Note from Figure 2.51 that the closed-loop settling time is about 19 seconds, while the maximum overshoot is about 8 percent. The closed-loop steady-state error is calculated to be $e(\infty) = 1 - \lim_{s \rightarrow 0} sY(s) = 0.1166$. Hence, the performance objectives have been met.

The robustness properties of the closed-loop chemical plant control system are indicated by a Bode plot of the closed-loop transfer function, which is shown along with the Bode plots of the plant and the lead compensator in Figure 2.52. Note that the lead compensator provides a phase lead and a gain increase in the frequency range $\alpha\omega_0 \leq \omega \leq \omega_0/\alpha$ (i.e. $0.0015 \leq \omega \leq 1500$ rad/s). This results in a *speeding-up* of the closed-loop response, which is evident in a reduced settling time and an increased phase crossover frequency. Using the CST command *margin*, the gain and phase margins of the plant are calculated to be 8.349 dB and 33.6° , respectively, with gain and phase crossover frequencies of 0.1957 rad/s and 0.3302 rad/s, respectively. The closed-loop system has, however, a gain margin of 32.79 dB and an *infinite* phase margin, with a phase crossover frequency of 2.691 rad/s. Hence, the closed-loop system has a greater robustness to transfer function variations, and a faster response than the plant. However, due to an increased closed-loop gain at high frequencies, the *roll-off* at high frequencies is reduced for the closed-loop system, when compared to the plant. This implies an increased sensitivity of the closed-loop system to the high-frequency measurement noise.

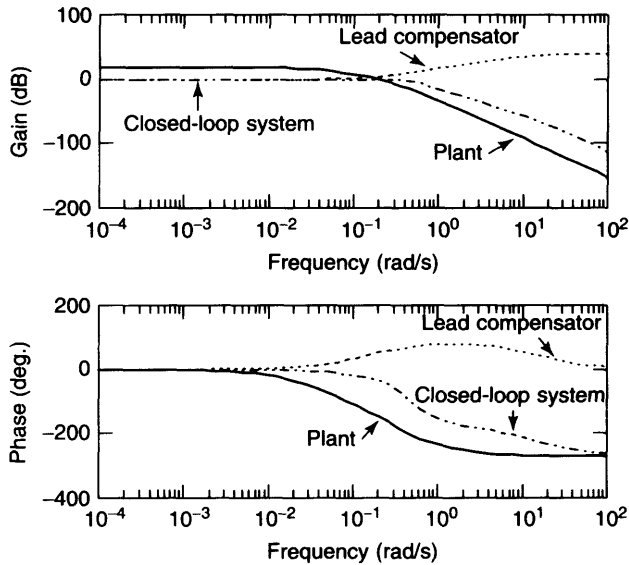


Figure 2.52 Bode plots of the large chemical plant, lead compensator, and the closed-loop system

From Examples 2.24 and 2.25, it is clear that while a lag compensator reduces the closed-loop steady-state error by providing a phase-lag at low frequencies, a lead compensator speeds-up the closed-loop response by providing a phase-lead at high frequencies. Let us see an application of the lead-lag compensation, which combines the desirable features of lag and lead compensators.

Example 2.26

In Example 2.25, the use of lead compensation resulted in a faster closed-loop response of a chemical plant. However, the closed-loop steady-state error with lead compensation was non-zero. Let us try to reduce the steady-state error, while speeding up the response, with the help of lead-lag compensation. In Example 2.25, a lead compensator with transfer function, $H(s) = (s + 0.15)/(0.01s + 0.15)$, produced a closed-loop steady-state error of 0.1166 and a settling time of 19 seconds. Let us choose the same transfer function for the lead part of the lag-lead compensator, i.e. $\omega_2 = 0.15$ rad/s and $1/\alpha = 0.01$ (i.e. $\alpha = 100$). The lag part of the lead-lag compensator would reduce the steady-state error by increasing the DC gain of the open-loop transfer function, $G(s)H(s)$. Let us therefore modify our design requirements to a closed-loop steady-state error of less than 0.002 and a settling time less than 20 seconds, while accepting a 10 percent maximum overshoot in the step response. To meet the new design requirements, a lag compensation frequency of $\omega_1 = 0.13$ rad/s is selected, resulting in the following lead-lag compensator transfer

function:

$$H(s) = [(s + 0.15)(s + 0.13)] / [(0.01s + 0.15)(s + 0.0013)] \quad (2.186)$$

which gives us the following closed-loop transfer function:

$$\begin{aligned} Y(s)/Y_d(s) \\ = (0.025s + 0.007) / (0.01s^4 + 0.15631s^3 + 0.095793s^2 + 0.041507s + 0.0075163) \end{aligned} \quad (2.187)$$

The closed-loop poles, damping, and natural frequencies are as follows:

```
>>nCL=[0.025 0.007];dCL=[1e-2 1.5631e-1 9.5793e-2 4.1507e-2 7.5163e-3];
damp(dCL) <enter>
```

Eigenvalue	Damping	Freq. (rad/sec)
-1.3483e-001+4.8777e-002i	9.4036e-001	1.4339e-001
-1.3483e-001-4.8777e-002i	9.4036e-001	1.4339e-001
-1.7514e-001+3.5707e-001i	4.4037e-001	3.9770e-001
-1.7514e-001-3.5707e-001i	4.4037e-001	3.9770e-001
-1.5011e+001	1.0000e+000	1.5011e+001

The first four poles, roughly located the same distance from the imaginary axis, are the dominant poles of the closed-loop system. The DC gain of the closed-loop system indicates the steady-state value of the step response, and is calculated using the MATLAB (CST) command *dcgain* as follows:

```
>>GCL=tf(nCL,dCL); dcgain(GCL) <enter>=
```

```
ans =
```

```
0.9987
```

Hence, the closed-loop steady-state error is brought down to $1 - 0.9987 = 0.0013$, which is acceptable. The closed-loop step response is plotted in Figure 2.53 for three different values of the lag frequency, ω_1 . For $\omega_1 = 0.13$ rad/s (used in the above calculations), the closed-loop response has a settling time of about 16 seconds, and a maximum overshoot of about 30 percent, which is unacceptable. Figure 2.53 shows that the maximum percentage overshoot can be reduced by decreasing ω_1 . For $\omega_1 = 0.05$ rad/s, the maximum percentage overshoot is about 10 percent, and the settling time is about 13 seconds. However, decreasing ω_1 below 0.05 rad/s results in an *increase* in the settling time, as is evident from the step response for $\omega_2 = 0.03$ rad/s which has a maximum overshoot of about 5 percent, but a settling time of about 22 seconds. The steady-state error is unaffected by changing ω_1 . We select $\omega_1 = 0.05$ rad/s, which gives a fast response and an acceptable maximum overshoot.

The Bode plots of the plant, the lead-lag compensator with $\omega_2 = 0.15$ rad/s, $\omega_1 = 0.05$ rad/s, $\alpha = 100$, and the resulting closed-loop system are shown in Figure 2.54.

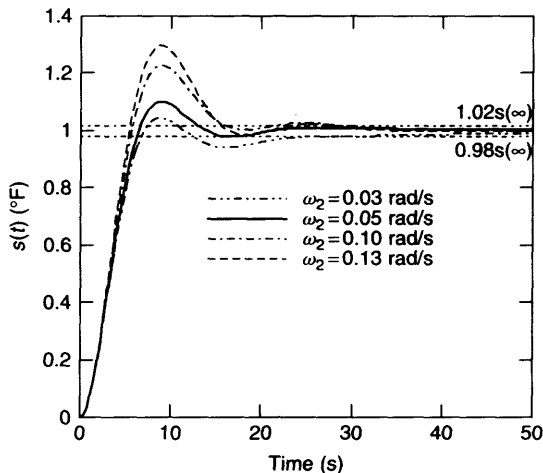


Figure 2.53 Closed-loop step response of the chemical plant with a lead-lag compensator for various values of the lag frequency, ω_2

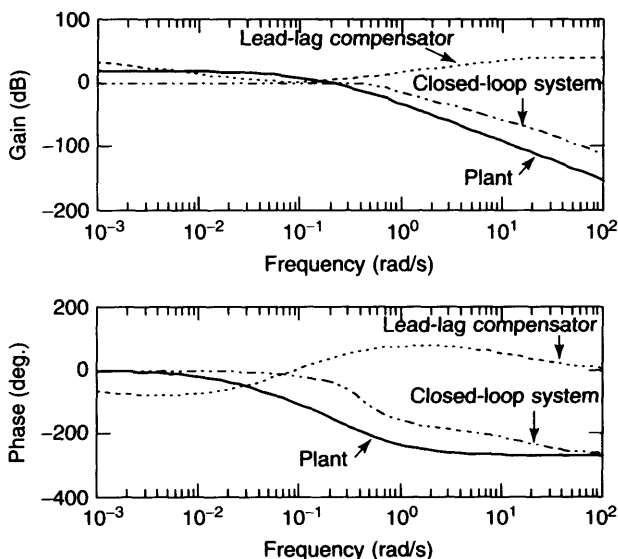


Figure 2.54 Bode plots of the large chemical plant, the lead-lag compensator, and the closed-loop system

Comparing Figure 2.54 with Figure 2.52, the phase-lag and decreasing compensator gain below 0.1 rad/s, and phase-lag and increasing compensator gain above 0.1 rad/s are evident for the lead-lag compensator. The high-frequency closed-loop behavior is largely unaffected, when compared to the lead compensation in Figure 2.52. The closed-loop gain and phase margins are computed by the command *margin* to be 38.9 dB and 104.5°, respectively, with gain and phase crossover frequencies of