2

Linear Systems and Classical Control

2.1 How Valid is the Assumption of Linearity?

It was mentioned in Chapter 1 that we need differential equations to describe the behavior of a system, and that the mathematical nature of the governing differential equations is another way of classifying control systems. In a large class of engineering applications, the governing differential equations can be assumed to be linear. The concept of linearity is one of the most important assumptions often employed in studying control systems. However, the following questions naturally arise: what is this assumption and how valid is it anyway? To answer these questions, let us consider lumped parameter systems for simplicity, even though all the arguments presented below are equally applicable to distributed systems. (Recall that *lumped parameter* systems are those systems whose behavior can be described by *ordinary* differential equations.) Furthermore, we shall confine our attention (until Section 2.13) to single-input, single-output (SISO) systems. For a general lumped parameter, SISO system (Figure 2.1) with input $u(t)$ and output $y(t)$, the governing ordinary differential equation can be written as

$$y^{(n)}(t) = f(y^{(n-1)}(t), y^{(n-2)}(t), \ldots, y^{(1)}(t), y(t), u^{(m)}(t), u^{(m-1)}(t), \ldots, u^{(1)}(t), u(t), t) \tag{2.1}$$

where $y^{(k)}$ denotes the kth derivative of $y(t)$ with respect to time, t, e.g. $y^{(n)} = d^n y/dt^n$, $y^{(n-1)} = d^{n-1}y/dt^{n-1}$, and $u^{(k)}$ denotes the kth time derivative of $u(t)$. This notation for derivatives of a function will be used throughout the book. In Eq. (2.1), $f()$ denotes a function of all the time derivatives of $y(t)$ of order $(n-1)$ and less, as well as the time derivatives of $u(t)$ of order m and less, and time, t. For most systems $m \leq n$, and such systems are said to be *proper*.

Since n is the order of the highest time derivative of $y(t)$ in Eq. (2.1), the system is said to be *of order n*. To determine the output $y(t)$, Eq. (2.1) must be somehow integrated in time, with $u(t)$ known and for specific initial conditions $y(0), y^{(1)}(0), y^{(2)}(0), \ldots, y^{(n-1)}(0)$. Suppose we are capable of solving Eq. (2.1), given any time varying input, $u(t)$, and the initial conditions. For simplicity, let us assume that the initial conditions are zero, and we apply an input, $u(t)$, which is a *linear combination* of two different inputs, $u_1(t)$, and $u_2(t)$, given by

$$u(t) = c_1u_1(t) + c_2u_2(t) \tag{2.2}$$

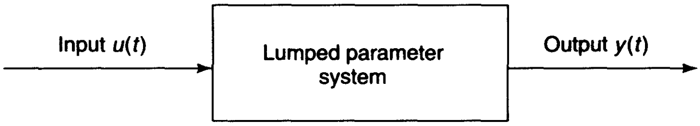

Figure 2.1 A general lumped parameter system with input, $u(t)$, and output, $y(t)$

where c_1 and c_2 are constants. If the resulting output, $y(t)$, can be written as

$$y(t) = c_1 y_1(t) + c_2 y_2(t) \tag{2.3}$$

where $y_1(t)$ is the output when $u_1(t)$ is the input, and $y_2(t)$ is the output when $u_2(t)$ is the input, then the system is said to be *linear*; otherwise it is called *nonlinear*. In short, a linear system is said to obey the *superposition principle*, which states that the output of a linear system to an input consisting of linear combination of two different inputs (Eq. (2.2)) can be obtained by linearly superposing the outputs to the respective inputs (Eq. (2.3)). (The superposition principle is also applicable for *non-zero* initial conditions, if the initial conditions on $y(t)$ and its time derivatives are linear combinations of the initial conditions on $y_1(t)$ and $y_2(t)$, and their corresponding time derivatives, with the constants c_1 and c_2.) Since linearity is a mathematical property of the governing differential equations, it is possible to say merely by inspecting the differential equation whether a system is linear. If the function $f()$ in Eq. (2.1) contains no powers (other than one) of $y(t)$ and its derivatives, or the mixed products of $y(t)$, its derivatives, and $u(t)$ and its derivatives, or transcendental functions of $y(t)$ and $u(t)$, then the system will obey the superposition principle, and its linear differential equation can be written as

$$\begin{aligned} &a_n y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \cdots + a_1 y^{(1)}(t) + a_0 y(t) \\ &\quad = b_m u^{(m)}(t) + b_{m-1} u^{(m-1)}(t) + \cdots + b_1 u^{(1)}(t) + b_0 u(t) \end{aligned} \tag{2.4}$$

Note that even though the coefficients $a_0, a_1, \ldots, a_n$ and $b_0, b_1, \ldots, b_m$ (called the *parameters* of a system) in Eq. (2.4) may be *varying* with time, the system given by Eq. (2.4) is still linear. A system with time-varying parameters is called a *time-varying* system, while a system whose parameters are constant with time is called *time-invariant* system. In the present chapter, we will be dealing only with linear, time-invariant systems. It is possible to express Eq. (2.4) as a *set* of lower order differential equations, whose individual orders add up to n. Hence, the order of a system is the *sum* of orders of all the differential equations needed to describe its behavior.

Example 2.1

For an electrical network shown in Figure 2.2, the governing differential equations are the following:

$$v_1^{(1)}(t) = -(v_1(t)/C_1)(1/R_1 + 1/R_3) + v_2(t)/(C_1 R_3) + e(t)/(R_1 C_1) \tag{2.5a}$$

$$v_2^{(1)}(t) = v_1(t)/(C_2 R_3) - (v_2(t)/C_2)(1/R_2 + 1/R_3) + e(t)/(R_2 C_2) \tag{2.5b}$$

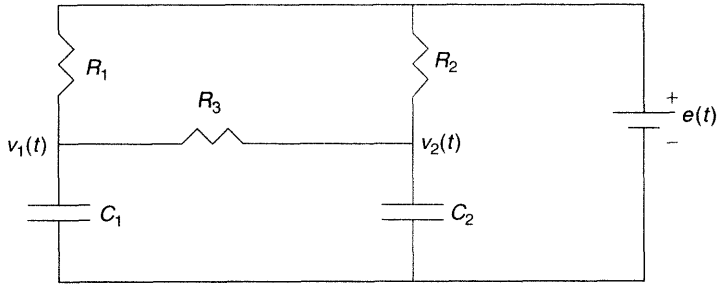

Figure 2.2 Electrical network for Example 2.1

where $v_1(t)$ and $v_2(t)$ are the voltages of the two capacitors, C_1 and C_2, $e(t)$ is the applied voltage, and R_1, R_2, and R_3 are the three resistances as shown.

On inspection of Eq. (2.5), we can see that the system is described by two *first order*, ordinary differential equations. Therefore, the system is of *second order*. Upon the substitution of Eq. (2.5b) into Eq. (2.5a), and by eliminating v_2, we get the following second order differential equation:

$$\begin{aligned} &C_1 R_3 v_1^{(2)}(t) + [R_3/R_1 + 1 + (C_1/C_2)(R_3/R_2 + 1)]v_1^{(1)}(t) \\ &\quad + [(1/R_2 + 1/R_3)(R_3/R_1 + 1) - 1/R_3]v_1(t) \\ &\quad = (R_3/R_1)(1/R_1 + 1/R_3)e(t)/C_2 + (R_3/R_1)e^{(1)}(t) \end{aligned} \tag{2.6}$$

Assuming $y(t) = v_1(t)$ and $u(t) = e(t)$, and comparing Eq. (2.6) with Eq. (2.4), we can see that there are no higher powers, transcendental functions, or mixed products of the output, input, and their time derivatives. Hence, *the system is linear*.

Suppose we *do not* have an input, $u(t)$, applied to the system in Figure 2.1. Such a system is called an *unforced system*. Substituting $u(t) = u^{(1)}(t) = u^{(2)}(t) = \cdots = u^{(m)}(t) = 0$ into Eq. (2.1) we can obtain the following governing differential equation for the unforced system:

$$y^{(n)}(t) = f(y^{(n-1)}(t), y^{(n-2)}(t), \ldots, y^{(1)}(t), y(t), 0, 0, \ldots, 0, 0, t) \tag{2.7}$$

In general, the solution, $y(t)$, to Eq. (2.7) for a given set of initial conditions is a function of time. However, there may also exist special solutions to Eq. (2.7) which are constant. Such constant solutions for an unforced system are called its *equilibrium points*, because the system continues to be at rest when it is already at such points. A large majority of control systems are designed for keeping a plant at one of its equilibrium points, such as the cruise-control system of a car and the autopilot of an airplane or missile, which keep the vehicle moving at a constant velocity. When a control system is designed for maintaining the plant at an equilibrium point, then only small deviations from the equilibrium point need to be considered for evaluating the performance of such a control system. Under such circumstances, the time behavior of the plant and the resulting control system can generally be assumed to be governed by linear differential equations, even though the governing differential equations of the plant and the control system may be nonlinear. The following examples demonstrate how a nonlinear system can be linearized near its equilibrium points. Also included is an example which illustrates that such a linearization may not always be possible.

Example 2.2

Consider a simple pendulum (Figure 2.3) consisting of a point mass, m, suspended from hinge at point O by a rigid massless link of length L. The equation of motion of the simple pendulum in the absence of an externally applied torque about point O in terms of the angular displacement, $\theta(t)$, can be written as

$$L\theta^{(2)}(t) + g.\sin(\theta(t)) = 0 \tag{2.8}$$

This governing equation indicates a second-order system. Due to the presence of $\sin(\theta)$ – a transcendental function of θ – Eq. (2.8) is nonlinear. From our everyday experience with a simple pendulum, it is clear that it can be brought to rest at only *two* positions, namely $\theta = 0$ and $\theta = \pi$ rad. (180°). Therefore, these two positions are the equilibrium points of the system given by Eq. (2.8). Let us examine the behavior of the system near each of these equilibrium points.

Since the only nonlinear term in Eq. (2.8) is $\sin(\theta)$, if we can show that $\sin(\theta)$ can be approximated by a linear term, then Eq. (2.8) can be linearized. Expanding $\sin(\theta)$ about the equilibrium point $\theta = 0$, we get the following *Taylor's series* expansion:

$$\sin(\theta) = \theta - \theta^3/3! + \theta^5/5! - \theta^7/7! + \cdots \tag{2.9}$$

If we assume that motion of the pendulum about $\theta = 0$ consists of small angular displacements (say $\theta < 10°$), then $\sin(\theta) \approx \theta$, and Eq. (2.8) becomes

$$L\theta^{(2)}(t) + g\theta(t) = 0 \tag{2.10}$$

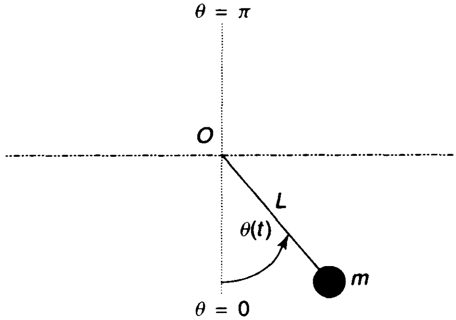

Figure 2.3 A simple pendulum of length L and mass m

Similarly, expanding $\sin(\theta)$ about the *other* equilibrium point, $\theta = \pi$, by assuming small angular displacement, ϕ, such that $\theta = \pi - \phi$, and noting that $\sin(\theta) = -\sin(\phi) \approx -\phi$, we can write Eq. (2.8) as

$$L\phi^{(2)}(t) - g\phi(t) = 0 \tag{2.11}$$

We can see that both Eqs. (2.10) and (2.11) are linear. Hence, the nonlinear system given by Eq. (2.8) has been linearized about both of its equilibrium points. Second order linear ordinary differential equations (especially the *homogeneous* ones like Eqs. (2.10) and (2.11)) can be be solved analytically. It is well known (and you may verify) that the solution to Eq. (2.10) is of the form $\theta(t) = \mathrm{A}.\sin(t(g/L)^{1/2} + \mathrm{B}.\cos(t(g/L)^{1/2})$, where the constants A and B are determined from the initial conditions, $\theta(0)$ and $\theta^{(1)}(0)$. This solution implies that $\theta(t)$ oscillates about the equilibrium point $\theta = 0$. However, the solution to Eq. (2.11) is of the form $\phi(t) = \mathrm{C}.\exp(t(g/L)^{1/2})$, where C is a constant, which indicates an exponentially increasing $\phi(t)$ if $\phi(0) \neq 0$. (This nature of the equilibrium point at $\theta = \pi$ can be experimentally verified by anybody trying to stand on one's head for any length of time!) The comparison of the solutions to the linearized governing equations close to the equilibrium points (Figure 2.4) brings us to an important property of an equilibrium point, called *stability*.

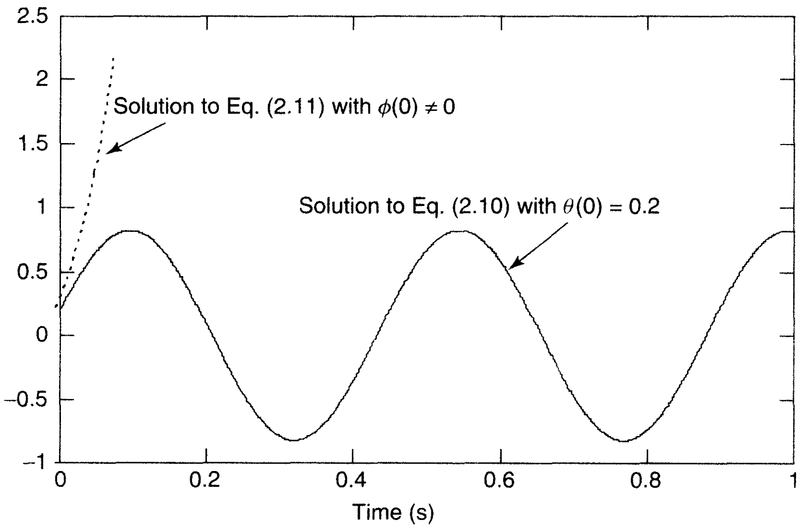

Figure 2.4 Solutions to the governing differential equation linearized about the two equilibrium points ($\theta = 0$ and $\theta = \pi$)

Stability is defined as the ability of a system to approach one of its equilibrium points once displaced from it. We will discuss stability in detail later. Here, suffice it to say that the pendulum is stable about the equilibrium point $\theta = 0$, but unstable about the equilibrium point $\theta = \pi$. While Example 2.2 showed how a nonlinear system can be linearized close to its equilibrium points, the following example illustrates how a nonlinear system's description can be transformed into a linear system description through a clever change of coordinates.

Example 2.3

Consider a satellite of mass m in an orbit about a planet of mass M (Figure 2.5). The distance of the satellite from the center of the planet is denoted $r(t)$, while its orientation with respect to the planet's equatorial plane is indicated by the angle $\theta(t)$, as shown. Assuming there are no gravitational anomalies that cause a departure from Newton's inverse-square law of gravitation, the governing equation of motion of the satellite can be written as

$$r^{(2)}(t) - h^2/r(t)^3 + k^2/r(t)^2 = 0 \tag{2.12}$$

where h is the constant angular momentum, given by

$$h = r(t)^2\theta^{(1)}(t) = \text{constant} \tag{2.13}$$

and $k = GM$, with G being the universal gravitational constant.

Equation (2.12) represents a nonlinear, second order system. However, since we are usually interested in the *path* (or the shape of the orbit) of the satellite, given by $r(\theta)$, rather than its distance from the planet's center as a function of time, r(t), we can transform Eq. (2.12) to the following linear differential equation by using the co-ordinate transformation $u(\theta) = 1/r(\theta)$:

$$u^{(2)}(\theta) + u(\theta) - k^2/h^2 = 0 \tag{2.14}$$

Being a linear, *second* order ordinary differential equation (similar to Eq. (2.10)), Eq. (2.14) is easily solved for $u(\theta)$, and the solution transformed back to $r(\theta)$

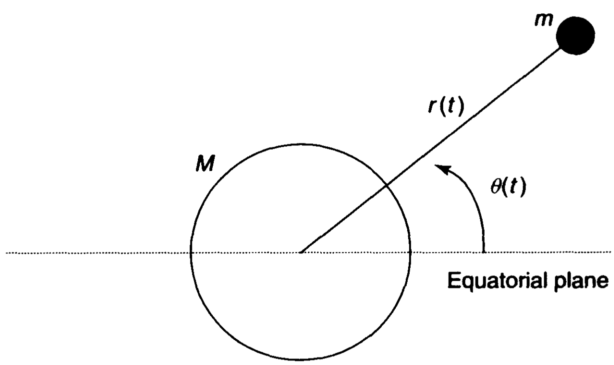

Figure 2.5 A satellite of mass m in orbit around a planet of mass M at a distance $r(t)$ from the planet's center, and azimuth angle $\theta(t)$ from the equatorial plane

given by

$$r(\theta) = (h^2/k^2)/[1 + A(h^2/k^2)\cos(\theta - \mathrm{B})] \tag{2.15}$$

where the constants A and B are determined from $r(\theta)$ and $r^{(1)}(\theta)$ specified at given values of θ. Such specifications are called *boundary conditions*, because they refer to points in space, as opposed to *initial conditions* when quantities at given instants of time are specified. Equation (2.15) can represent a circle, an ellipse, a parabola, or a hyperbola, depending upon the magnitude of $\mathrm{A}(h^2/k^2)$ (called the *eccentricity* of the orbit).

Note that we could also have linearized Eq. (2.12) about one of its equilibrium points, as we did in Example 2.2. One such equilibrium point is given by $r(t) =$ constant, which represents a circular orbit. Many practical orbit control applications consist of minimizing deviations from a given circular orbit using rocket thrusters to provide *radial acceleration* (i.e. acceleration along the line joining the satellite and the planet) as an input, $u(t)$, which is based upon the measured deviation from the circular path fed back to an onboard controller, as shown in Figure 2.6. In such a case, the governing differential equation is no longer homogeneous as Eq. (2.12), but has a *non-homogeneous* forcing term on the right-hand side given by

$$r^{(2)}(t) - h^2/r(t)^3 + k^2/r(t)^2 = u(t) \tag{2.16}$$

Since the deviations from a given circular orbit are usually small, Eq. (2.16) can be suitably linearized about the equilibrium point $r(t) = \mathrm{C}$. (This linearization is left as an exercise for you at the end of the chapter.)

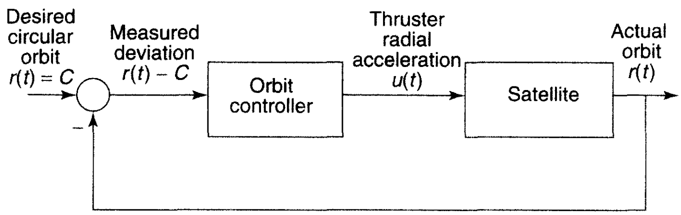

Figure 2.6 On orbit feedback control system for maintaining a circular orbit of a satellite around a planet

Examples 2.2 and 2.3 illustrated how a nonlinear system can be linearized for practical control applications. However, as pointed out earlier, it is not always possible to do so. If a nonlinear system has to be moved from one equilibrium point to another (such as changing the speed or altitude of a cruising airplane), the assumption of linearity that is possible in the close neighborhood of each equilibrium point disappears as we cross the nonlinear region between the equilibrium points. Also, if the motion of a nonlinear system consists of large deviations from an equilibrium point, again the concept of linearity is not valid. Lastly, the characteristics of a nonlinear system may be such that it does not have any equilibrium point about which it can be linearized. The following missile guidance example illustrates such a nonlinear system.

Example 2.4

Radar or laser-guided missiles used in modern warfare employ a special guidance scheme which aims at flying the missile along a radar or laser beam that is illuminating a moving target. The guidance strategy is such that a correcting command signal (input) is provided to the missile if its flight path deviates from the moving beam. For simplicity, let us assume that both the missile and the target are moving in the same plane (Figure 2.7). Although the distance from the beam source to the target, $R_T(t)$, is not known, it is assumed that the angles made by the missile and the target with respect to the beam source, $\theta_M(t)$ and $\theta_T(t)$, are available for precise measurement. In addition, the distance of the missile from the beam source, $R_M(t)$, is also known at each instant.

A *guidance law* provides the following normal acceleration command signal, $a_c(t)$, to the missile

$$a_c(t) = K R_M(t)[\theta_T(t) - \theta_M(t)] \tag{2.17}$$

As the missile is usually faster than the target, if the angular deviation $[\theta_T(t) - \theta_M(t)]$ is made small enough, the missile will intercept the target. The feedback guidance scheme of Eq. (2.17) is called *beam-rider guidance*, and is shown in Figure 2.8.

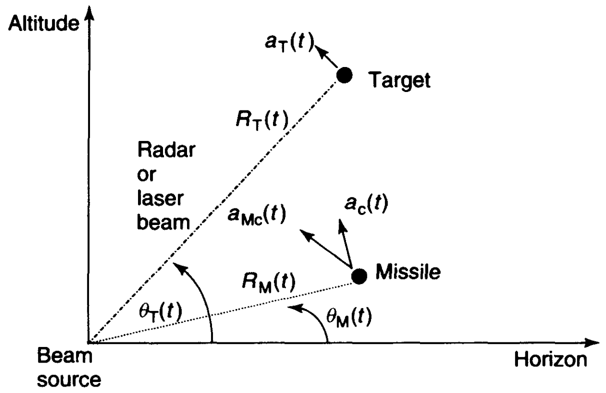

Figure 2.7 Beam guided missile follows a beam that continuously illuminates a moving target located at distance $R_T(t)$ from the beam source

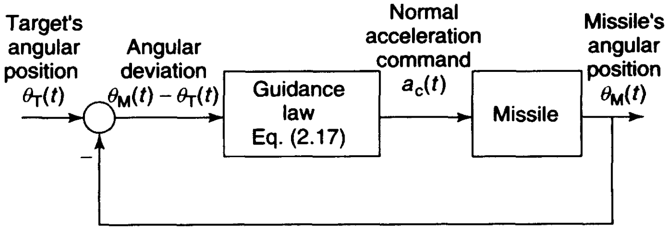

Figure 2.8 Beam-rider closed-loop guidance for a missile

The beam-rider guidance can be significantly improved in performance if we can measure the angular velocity, $\theta_T^{(1)}(t)$, and the angular acceleration, $\theta_T^{(2)}(t)$, of the target. Then the beam's normal acceleration can be determined from the following equation:

$$a_T = R_T(t)\theta_T^{(2)}(t) + 2R_T^{(1)}(t)\theta_T^{(1)}(t) \tag{2.18}$$

In such a case, along with $a_c(t)$ given by Eq. (2.17), an additional command signal (input) can be provided to the missile in the form of missile's acceleration perpendicular to the beam, $a_{Mc}(t)$, given by

$$a_{Mc}(t) = R_M(t)\theta_M^{(2)}(t) + 2R_M^{(1)}(t)\theta_M^{(1)}(t) \tag{2.19}$$

Since the final objective is to make the missile intercept the target, it must be ensured that $\theta_M^{(1)}(t) = \theta_T^{(1)}(t)$ and $\theta_M^{(2)}(t) = \theta_T^{(2)}(t)$, even though $[\theta_T(t) - \theta_M(t)]$ may not be *exactly* zero. (To understand this philosophy, remember how we catch up with a friend's car so that we can chat with her. We accelerate (or decelerate) until our velocity (and acceleration) become identical with our friend's car, then we can talk with her; although the two cars are abreast, *they are not exactly in the same position.*) Hence, the following command signal for missile's normal acceleration perpendicular to the beam must be provided:

$$a_{Mc}(t) = R_M(t)\theta_T^{(2)}(t) + 2R_M^{(1)}(t)\theta_T^{(1)}(t) \tag{2.20}$$

The guidance law given by Eq. (2.20) is called *command line-of-sight guidance*, and its implementation along with the beam-rider guidance is shown in the block diagram of Figure 2.9. It can be seen in Figure 2.9 that while $\theta_T(t)$ is being fed back, the angular velocity and acceleration of the target, $\theta_T^{(1)}(t)$, and $\theta_T^{(2)}(t)$, respectively, are being fed forward to the controller. Hence, similar to the control system of Figure 1.4, additional information about the target is being provided by a feedforward loop to improve the closed-loop performance of the missile guidance system.

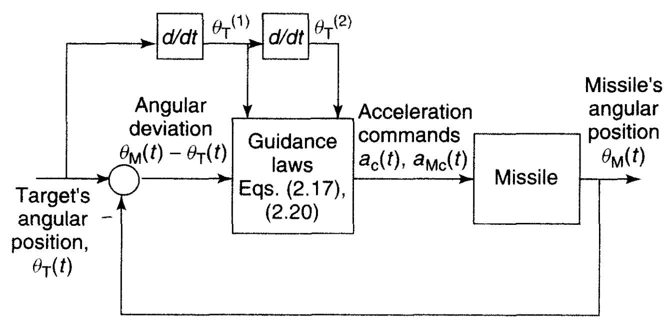

Figure 2.9 Beam-rider and command line-of-sight guidance for a missile

Note that both Eq. (2.17) and Eq. (2.20) are nonlinear in nature, and generally cannot be linearized about an equilibrium point. This example shows that the concept of linearity