

is not always valid. For more information on missile guidance strategies, you may refer to the excellent book by Zarchan [1].

2.2 Singularity Functions

It was mentioned briefly in Chapter 1 that some peculiar, well known input functions are generally applied to test the behavior of an unknown system. A set of such test functions is called *singularity functions*. The singularity functions are important because they can be used as building blocks to construct any arbitrary input function and, by the superposition principle (Eq. (2.3)), the response of a linear system to any arbitrary input can be easily obtained as the linear superposition of responses to singularity functions. The two distinct singularity functions commonly used for determining an unknown system's behavior are the *unit impulse* and *unit step* functions. A common property of these functions is that they are continuous in time, *except at a given time*. Another interesting fact about the singularity functions is that they can be derived from each other by differentiation or integration in time.

The unit impulse function (also called the *Dirac delta function*), $\delta(t - a)$, is seen in Figure 2.10 to be a very large spike occurring for a very small duration, applied at time $t = a$, such that the total area under the curve (shaded region) is unity. A unit impulse function can be multiplied by a constant to give a general impulse function (whose area under the curve is not unity). From this description, we recognize an impulse function to be the force one feels when hit by a car – and in all other kinds of impacts.

The height of the rectangular pulse in Figure 2.10 is $1/\varepsilon$, whereas its width is ε seconds, ε being a very small number. In the limit $\varepsilon \rightarrow 0$, the unit impulse function tends to infinity (i.e. $\delta(t - a) \rightarrow \infty$). The unit impulse function shown in Figure 2.10 is an idealization of the actual impulse whose shape is not rectangular, because it takes some time to reach the maximum value, unlike the unit impulse function (which becomes very large instantaneously). Mathematically, the unit impulse function can be described by the following equations:

$$\delta(t - a) = 0, \quad \text{for } t \neq a \quad (2.21)$$

$$\int_{-\infty}^{\infty} \delta(t - a) dt = 1 \quad (2.22)$$

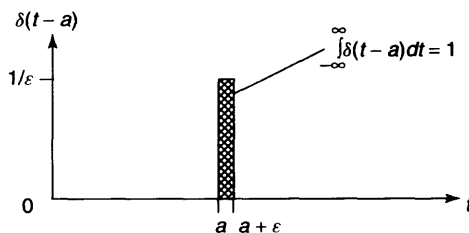


Figure 2.10 The unit impulse function; a pulse of infinitesimal duration (ε) and very large magnitude ($1/\varepsilon$) such that its total area is unity

Note that $\delta(t - a)$ is discontinuous at $t = a$. Furthermore, since the unit impulse function is non-zero only in the period $a \leq t \leq a + \varepsilon$, we can also express Eqs. (2.21) and (2.22) by

$$\int_a^{a+\varepsilon} \delta(t - a) dt = 1 \quad (2.23)$$

However, when utilizing the unit impulse function for control applications, Eq. (2.22) is much more useful. In fact, if $\delta(t - a)$ appears inside an integral with infinite integration limits, then such an integral is very easily carried out with the use of Eqs. (2.21) and (2.22). For example, if $f(t)$ is a continuous function, then the well known *Mean Value Theorem* of integral calculus can be applied to show that

$$\int_{T_1}^{T_2} f(t) \delta(t - a) dt = f(a) \int_{T_1}^{T_2} \delta(t - a) dt = f(a) \quad (2.24)$$

where $T_1 < a < T_2$. Equation (2.24) indicates an important property of the unit impulse function called the *sampling property*, which allows the time integral of any continuous function $f(t)$ weighted by $\delta(t - a)$ to be simply equal to the function $f(t)$ evaluated at $t = a$, provided the limits of integration bracket the time $t = a$.

The unit step function, $u_s(t - a)$, is shown in Figure 2.11 to be a jump of unit magnitude at time $t = a$. It is aptly named, because it resembles a step of a staircase. Like the unit impulse function, the unit step function is also a mathematical idealization, because it is impossible to apply a non-zero input instantaneously. Mathematically, the unit step function can be defined as follows:

$$u_s(t - a) = \begin{cases} 0 & \text{for } t < a \\ 1 & \text{for } t > a \end{cases} \quad (2.25)$$

It is clear that $u_s(t - a)$ is discontinuous at $t = a$, and its time derivative at $t = a$ is infinite. Recalling from Figure 2.10 that in the limit $\varepsilon \rightarrow 0$, the unit impulse function tends to infinity (i.e. $\delta(t - a) \rightarrow \infty$), we can express the unit impulse function, $\delta(t - a)$, as the time derivative of the unit step function, $u_s(t - a)$, at time $t = a$. Also, since the time derivative of $u_s(t - a)$ is zero at all times, except at $t = a$ (where it is infinite), we can write

$$\delta(t - a) = du_s(t - a)/dt \quad (2.26)$$

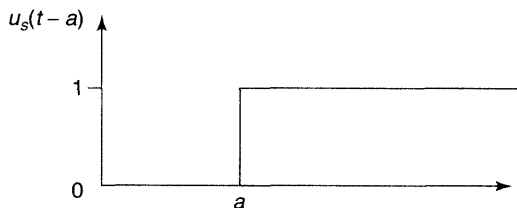


Figure 2.11 The unit step function, $u_s(t - a)$; a jump of unit magnitude at time $t = a$

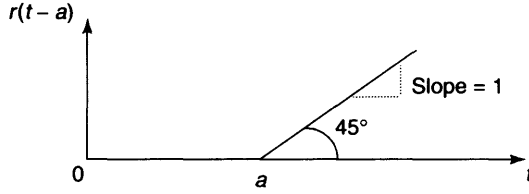


Figure 2.12 The unit ramp function; a ramp of unit slope applied at time $t = a$

Or, conversely, the unit step function is the time integral of the unit impulse function, given by

$$u_s(t - a) = \int_{-\infty}^t \delta(\tau - a) d\tau \quad (2.27)$$

A useful function related to the unit step function is the *unit ramp* function, $r(t - a)$, which is seen in Figure 2.12 to be a ramp of unit slope applied at time $t = a$. It is like an upslope of 45° angle you suddenly encounter while driving down a perfectly flat highway at $t = a$. Mathematically, $r(t - a)$ is given by

$$r(t - a) = \begin{cases} 0 & \text{for } t < a \\ (t - a) & \text{for } t > a \end{cases} \quad (2.28)$$

Note that $r(t - a)$ is continuous everywhere, but its slope is discontinuous at $t = a$. Comparing Eq. (2.28) with Eq. (2.25), it is clear that

$$r(t - a) = (t - a)u_s(t - a) \quad (2.29)$$

or

$$r(t - a) = \int_{-\infty}^t u_s(\tau - a) d\tau \quad (2.30)$$

Thus, the unit ramp function is the *time integral* of the unit step function, or conversely, the unit step function is the time derivative of the unit ramp function, given by

$$u_s(t - a) = dr(t - a)/dt \quad (2.31)$$

The basic singularity functions (unit impulse and step), and their relatives (unit ramp function) can be used to synthesize more complicated functions, as illustrated by the following examples.

Example 2.5

The *rectangular pulse* function, $f(t)$, shown in Figure 2.13, can be expressed by subtracting one step function from another as

$$f(t) = f_o[u_s(t + T/2) - u_s(t - T/2)] \quad (2.32)$$

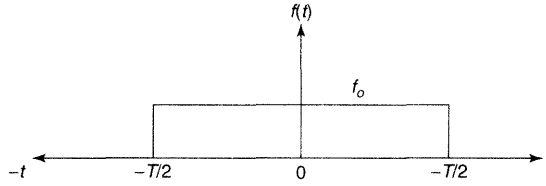


Figure 2.13 The rectangular pulse function of magnitude f_o

Example 2.6

The *decaying exponential function*, $f(t)$ (Figure 2.14) is zero before $t = 0$, and decays exponentially from a magnitude of f_o at $t = 0$. It can be expressed by multiplying the unit step function with f_o and a decaying exponential term, given by

$$f(t) = f_o e^{-t/\tau} u_s(t) \quad (2.33)$$

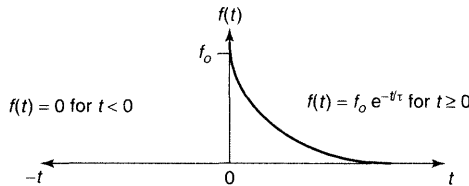


Figure 2.14 The decaying exponential function of magnitude f_o

Example 2.7

The *sawtooth pulse function*, $f(t)$, shown in Figure 2.15, can be expressed in terms of the unit step and unit ramp functions as follows:

$$f(t) = (f_o/T)[r(t) - r(t - T)] - f_o u_s(t - T) \quad (2.34)$$

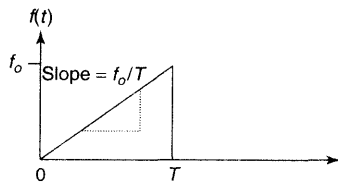


Figure 2.15 The sawtooth pulse of height f_o and width T

After going through Examples 2.5–2.7, and with a little practice, you can decide merely by looking at a given function how to synthesize it using the singularity functions. The unit impulse function has a special place among the singularity functions, because it can be