90° radial lines represent the real and imaginary parts, respectively, of $G(i\omega)$. The polar curve is seen in Figure 2.21 to be a circle of radius 0.00835 centered at the point 0.00835 on the real axis. The direction of increasing $\omega$ is shown by arrows on the polar curve. The shape and direction (with increasing $\omega$) of a polar plot gives valuable insight about a linear system's stability, which will be seen in Section 2.10.

## 2.4 Laplace Transform and the Transfer Function

In the previous section we had confined our attention to the *steady-state* response of a linear system to harmonic inputs. Here we would like to consider the *total* response (both transient and steady-state) of a linear, single-input, single-output system when the applied input is some *arbitrary* function of time. We saw how the representation of a harmonic input by a complex function *transformed* the governing differential equations into a complex algebraic expression for the frequency response. For a general input, a similar complex expression can be obtained by applying the *Laplace transformation* (denoted by $\mathcal{L}$) to the input, $u(t)$, defined as

$$U(s) = \mathcal{L}u(t) = \int_0^\infty e^{-st}u(t)\,dt \tag{2.59}$$

where $s$ denotes the *Laplace variable* (a complex number), and $U(s)$ is called the *Laplace transform* of $u(t)$. The Laplace transform of a function $u(t)$ is defined only if the infinite integral in Eq. (2.59) exists, and converges to a functional form, $U(s)$. However, if $U(s)$ exists, then it is *unique*. The convergence of the Laplace integral depends solely upon the shape of the function, $u(t)$. It can be shown rigourously that the Laplace integral converges only if $u(t)$ is *piecewise continuous* (i.e. any time interval, however large, can be broken up into a finite number of sub-intervals over each of which $u(t)$ is continuous, and at the ends of each sub-interval, $u(t)$ is finite) and *bounded by an exponential* (i.e. there exists a constant $a$ such that $e^{-at}|u(t)|$ is *bounded* at all times). The term *bounded* implies that a function's value lies between two *finite* limits. Most of the commonly used input functions are Laplace transformable. For example, if $u(t)$, is *discontinuous* (i.e. it has a jump) at $t = 0$, such as $u(t) = \delta(t)$ or $u(t) = u_s(t)$, we can obtain its Laplace transform. In such a case, the lower limit of integration in Eq. (2.59) is understood to be *just before* $t = 0$, i.e. just prior to the discontinuity in $u(t)$. Some important properties of the Laplace transform are stated below, and you may verify each of them using the definition given by Eq. (2.59):

(a) Linearity:
If $a$ is a constant (or independent of $s$ and $t$) and $\mathcal{L}f(t) = F(s)$, then

$$\mathcal{L}\{af(t)\} = a\mathcal{L}f(t) = aF(s) \tag{2.60}$$

Also, if $\mathcal{L}f_1(t) = F_1(s)$ and $\mathcal{L}f_2(t) = F_2(s)$, then

$$\mathcal{L}\{f_1(t) + f_2(t)\} = F_1(s) + F_2(s) \tag{2.61}$$

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(b) Complex differentiation:
If $\mathcal{L}f(t) = F(s)$, then

$$\mathcal{L}\{tf(t)\} = -dF(s)/ds \tag{2.62}$$

(c) Complex integration:
If $\mathcal{L}f(t) = F(s)$, and if $\lim_{t\to 0} f(t)/t$ exists as $t = 0$ is approached from the *positive side*, then

$$\mathcal{L}\{f(t)/t\} = \int_s^{\infty} F(s)\,ds \tag{2.63}$$

(d) Translation in time:
If $\mathcal{L}f(t) = F(s)$, and a is a positive, real number such that $f(t-a) = 0$ for $0 < t < a$, then

$$\mathcal{L}f(t-a) = \mathrm{e}^{-as}F(s) \tag{2.64}$$

(e) Translation in Laplace domain:
If $\mathcal{L}f(t) = F(s)$, and $a$ is a complex number, then

$$\mathcal{L}\{\mathrm{e}^{at}f(t)\} = F(s-a) \tag{2.65}$$

(f) Real differentiation:
If $\mathcal{L}f(t) = F(s)$, and if $f^{(1)}(t)$ is Laplace transformable, then

$$\mathcal{L}f^{(1)}(t) = sF(s) - f(0^+) \tag{2.66}$$

where $f(0^+)$ denotes the value of $f(t)$ in the limit $t \to 0$, approaching $t = 0$ from the *positive side*. If we apply the real differentiation property successively to the higher order time derivatives of $f(t)$ (assuming they are Laplace transformable), we can write the Laplace transform of the $k$th derivative, $f^{(k)}(t)$, as follows:

$$\mathcal{L}f^{(k)}(t) = s^kF(s) - s^{k-1}f(0^+) - s^{k-2}f^{(1)}(0^+) - \cdots - f^{(k-1)}(0^+) \tag{2.67}$$

(g) Real integration:
If $\mathcal{L}f(t) = F(s)$, and the indefinite integral $\int f(t)\,dt$ is Laplace transformable, then

$$\mathcal{L}\left\{\int f(t)\,dt\right\} = F(s)/s + (1/s)\int_{-\infty}^{0} f(t)\,dt \tag{2.68}$$

Note that the integral term on the right-hand side of Eq. (2.68) is zero if $f(t) = 0$ for $t < 0$.

(h) Initial value theorem:
If $\mathcal{L}f(t) = F(s)$, $f^{(1)}(t)$ is Laplace transformable, and $\lim_{s\to\infty} sF(s)$ exists, then

$$f(0^+) = \lim_{s\to\infty} sF(s) \tag{2.69}$$

(i) Final value theorem:
If $\mathcal{L}f(t) = F(s)$, $f^{(1)}(t)$ is Laplace transformable, and $\lim_{t\to\infty} f(t) = f(\infty)$ exists, then

$$f(\infty) = \lim_{s\to 0} sF(s) \tag{2.70}$$

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Since we are usually dealing with positive values of time, we will replace $0^+$ by 0 in all relevant applications of the Laplace transform. It is easy to see that if the input, $u(t)$, and its time derivatives are Laplace transformable, then the differential equation (Eq. (2.4)) of a linear, *time-invariant* system is Laplace transformable, which implies that the output, $y(t)$, is also Laplace transformable, whose Laplace transform is $Y(s)$. For simplicity, we assume that all initial conditions for the input, $u(t)$, and its derivatives and the output, $y(t)$, and its derivatives are zeros. Then, using Eq. (2.67) we can transform the governing equation of the system (Eq. (2.4)) to the Laplace domain as follows:

$$(s^n a_n + s^{n-1}a_{n-1} + \cdots + sa_1 + a_o)Y(s) = (s^m b_m + s^{m-1}b_{m-1} + \cdots + sb_1 + b_o)U(s) \tag{2.71}$$

Equation (2.71) brings us to one of the most important concepts in control theory, namely the *transfer function*, $G(s)$, which is defined as the ratio of the Laplace transform of the output, $Y(s)$, and that of the input, $U(s)$, given by

$$G(s) = Y(s)/U(s) \tag{2.72}$$

Substituting Eq. (2.71) into (2.72), we obtain the following expression for the transfer function of a linear, single-input, single-output system:

$$G(s) = (s^m b_m + s^{m-1}b_{m-1} + \cdots + sb_1 + b_o)/(s^n a_n + s^{n-1}a_{n-1} + \cdots + sa_1 + a_o) \tag{2.73}$$

As we saw in Chapter 1, the transfer function, $G(s)$, represents how an input, $U(s)$, is *transferred* to the output, $Y(s)$, or, in other words, the *relationship* between the input and output, when the initial conditions are *zero*. The transfer function representation of a system is widely used in block diagrams, such as Figure 2.22, and is very useful for even such systems for which the governing differential equations are not available. For such unknown systems, the transfer function is like a *black-box* defining the system's characteristics.

By applying known inputs (such as the singularity functions or harmonic signals) and measuring the output, one can determine an unknown system's transfer function experimentally. To do so, we have to see what are the relationships between the transfer function and the responses to singularity functions, and between the transfer function and the frequency response. The latter relationship is easily obtained by comparing Eq. (2.73) defining the transfer function, $G(s)$, with Eq. (2.47), which defines the frequency response, $G(i\omega)$. We see that the two quantities can be obtained from one another by using the relationship $s = i\omega$ (that is the reason why we knowingly used the same symbol, $G(\cdot)$, for both transfer function and the frequency response). A special transform, called the *Fourier transform*, can be defined by substituting $s = i\omega$ in the definition of the Laplace transform (Eq. (2.59). Fourier transform is widely used as a method of calculating the

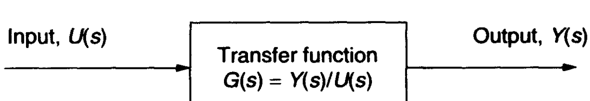


**Figure 2.22** Transfer function representation of a single-input, single-output system

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response of linear systems to arbitrary inputs by transforming an arbitrary input, $u(t)$, to its *frequency domain* counterpart, $U(i\omega)$ as follows:

$$U(i\omega) = \int_0^\infty e^{-i\omega t} u(t)\, dt \tag{2.74}$$

(The lower limit of integration in Eq. (2.74) is replaced by $-\infty$ if $u(t) \neq 0$ for $t < 0$.) Then, from Eq. (2.72), we can determine the resulting output (assuming zero initial conditions) in the frequency domain as $Y(i\omega) = G(i\omega)U(i\omega)$ (where $G(i\omega)$ is the pre-determined frequency response), and apply the *inverse Fourier transform* to obtain the output in the time-domain as follows:

$$y(t) = 1/(2\pi) \int_{-\infty}^{\infty} e^{i\omega t} Y(i\omega)\, d\omega \tag{2.75}$$

Note that in Eqs. (2.74) and (2.75), the Fourier transforms of the input and the output, $U(i\omega)$ and $Y(i\omega)$, do not have any physical significance, and in this respect they are similar to the Laplace transforms, $U(s)$ and $Y(s)$. However, the frequency response, $G(i\omega)$, is related to the steady-state response to harmonic input (as seen in Section 2.3), and can be experimentally measured. The transfer function, $G(s)$, however, is a useful mathematical abstraction, and cannot be experimentally measured in the Laplace domain. The Laplace variable, $s$, is a complex quantity, $s = \sigma \pm i\omega$, whose real part, $\sigma$, denotes whether the amplitude of the input (or output) is increasing or decreasing with time. We can grasp this fact by applying the *inverse Laplace transform*, $\mathcal{L}^{-1}$ (i.e. going from the Laplace domain to the time domain) to Eq. (2.59)

$$y(t) = \mathcal{L}^{-1} Y(s) = 1/(2\pi i) \int_{\sigma - i\infty}^{\sigma + i\infty} Y(s) e^{st} ds \tag{2.76}$$

where the integral is performed along an *infinitely* long line, parallel to the imaginary axis with a constant real part, $\sigma$ (Figure 2.23). Note that inverse Laplace transform is possible, because $Y(s)$ (if it exists) is unique.

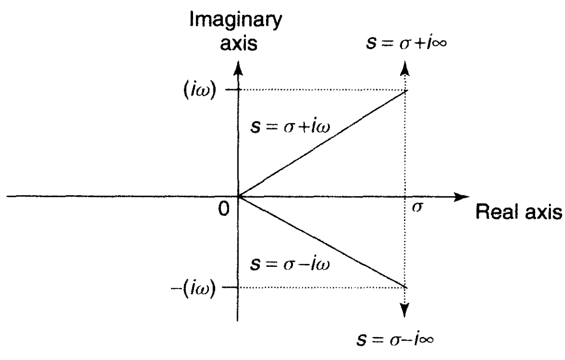


**Figure 2.23** The Laplace domain

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From Eq. (2.76) we can see that a general output, $y(t)$, will consist of such terms as $y_0 e^{st}$ (where $y_0$ is a *constant*), which can be expressed as $y_0 e^{\sigma t} e^{\pm i\omega t}$. The latter term indicates a periodically changing quantity of frequency, $\omega$, whose amplitude is a function of time given by $y_0 e^{\sigma t}$. When dealing with non-harmonic inputs and outputs, the use of the Laplace transform and the transfer function, $G(s)$, is more rewarding than working with the Fourier transform and the frequency response, $G(i\omega)$, because the resulting algebraic expressions are much simpler through the use of $s$ rather than $(i\omega)$. However, use of $G(s)$ involves interpreting system characteristics from complex (rather than purely imaginary) numbers.

The roots of the numerator and denominator polynomials of the transfer function, $G(s)$, given by Eq. (2.73) represent the characteristics of the linear, time-invariant system. The *denominator* polynomial of the transfer function, $G(s)$, equated to zero is called the *characteristic equation* of the system, given by

$$s^n a_n + s^{n-1} a_{n-1} + \cdots + s a_1 + a_0 = 0 \tag{2.77}$$

The roots of the characteristic equation are called the *poles* of the system. The roots of the *numerator* polynomial of $G(s)$ equated to zero are called the *zeros* of the transfer function, given by

$$s^m b_m + s^{m-1} b_{m-1} + \cdots + s b_1 + b_0 = 0 \tag{2.78}$$

In terms of its poles and zeros, a transfer function can be represented as a ratio of *factorized* numerator and denominator polynomials, given by the following *rational expression*:

$$\begin{aligned} G(s) &= K(s - z_1)(s - z_2)\ldots(s - z_m)/[(s - p_1)(s - p_2)\ldots(s - p_n)] \\ &= K\prod_{i=1}^{m}(s - z_i)/\prod_{j=1}^{n}(s - p_j) \end{aligned} \tag{2.79}$$

where $K$ is a constant (sometimes referred to as the *gain*), $z_i (i = 1, 2, \ldots, m)$ and $p_j (j = 1, 2, \ldots, n)$ are the zeros and poles of the system, respectively, and $\Pi$ is a short-hand notation denoting a *product* of many terms (in the same manner as $\Sigma$ denotes a *summation* of many terms). Equation (2.79) is also called *zero-pole-gain* description of a linear, time-invariant system, which can be modeled by the MATLAB Control System Toolbox's (CST) *LTI* object, *zpk*. As in Eq. (2.1), we repeat that for most linear, time-invariant systems $m \leq n$. Such systems are said to be *proper*. If $m < n$, the system is said to be *strictly proper*. Also, note that some zeros, $z_i$, and poles, $p_j$, may be *repeated* (i.e. two or more poles (or zeros) having identical values). Such a pole (or zero) is said to be *multiple*, and its *degree of multiplicity* is defined as the number of times it occurs. Finally, it may happen for some systems that a pole has the same value as a zero (i.e. $p_j = z_i$ for some pair $(i,j)$). Then the transfer function representation of Eq. (2.79) will not contain those poles and zeros, because they have canceled each other out. Pole-zero cancelations have a great impact on a system's *controllabilty* or *observability* (which will be studied in Chapter 5).

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### *Example 2.9*

Revisiting the electrical network of Example 2.8, we can write the system's transfer function as

$$G(s) = 0.5s/(s^2 + 30s + 10^6) \tag{2.80}$$

which indicates a zero at the origin ($z_1 = 0$), and the two complex poles given by the solution of the following quadratic characteristic equation:

$$s^2 + 30s + 10^6 = 0 \tag{2.81}$$

To get a better insight into the characteristics of a system, we can express *each* quadratic factor (such as that on the left-hand side of Eq. (2.81)) of the denominator polynomial as $s^2 + 2\varsigma\omega_n s + \omega_n^2$, where $\omega_n$ is a *natural frequency* of the system (see Section 2.3), and $\varsigma$ is called the *damping ratio.* The damping ratio, $\varsigma$, governs how rapidly the *magnitude* of the response of an *unforced* system decays with time. For a mechanical or electrical system, *damping* is the property which converts a part of the unforced system's *energy* to heat, thereby causing the system's energy – and consequently the output – to dissipate with time. Examples of damping are resistances in electrical circuits and friction in mechanical systems. From the discussion following Eq. (2.76), it can be seen that $\varsigma$ is closely related to the real part, $\sigma$, of a complex root of the characteristic equation (pole) given by $s = \sigma \pm i\omega$. The roots of the characteristic equation (or, in other words, the poles of the system) expressed as

$$s^2 + 2\varsigma\omega_n s + \omega_n^2 = 0 \tag{2.82}$$

are

$$s = p_1 = -\varsigma\omega_n - i\omega_n(\varsigma^2 - 1)^{1/2} \tag{2.83}$$

and

$$s = p_2 = -\varsigma\omega_n + i\omega_n(\varsigma^2 - 1)^{1/2} \tag{2.84}$$

Note that the real part of each pole is $\sigma = -\varsigma\omega_n$, while the imaginary parts are $\pm\omega = \pm\omega_n(\varsigma^2 - 1)^{1/2}$. For the present example, the poles are found by solving Eq. (2.81) to be $p_{1,2} = -15 \pm 999.9i$, which implies that the natural frequency and damping-ratio are, $\omega_n = 1000$ rad/s and $\varsigma = 0.015$, respectively. These numbers could also have been obtained by comparing Eq. (2.81) and Eq. (2.82). The natural frequency agrees with our calculation in Example 2.8, which was also observed as a peak in the Bode gain plot of Figure 2.20. The positive damping-ratio (or the negative real part of the complex poles) indicates that the amplitude of the response to any input will decay with time due to the presence of terms such as $y_0 e^{\sigma t} e^{\pm i\omega t}$ in the expression for the output, $y(t)$.

One can see the dependence of the response, $y(t)$, on the damping-ratio, $\varsigma$, in Figure 2.24, which is a plot of a typical initial response of an unforced second order system. $\varsigma = 1$ is the limiting case, called *critical damping*, because it denotes the boundary between *oscillatory* and *exponentially decaying* response. For $0 < \varsigma < 1$,

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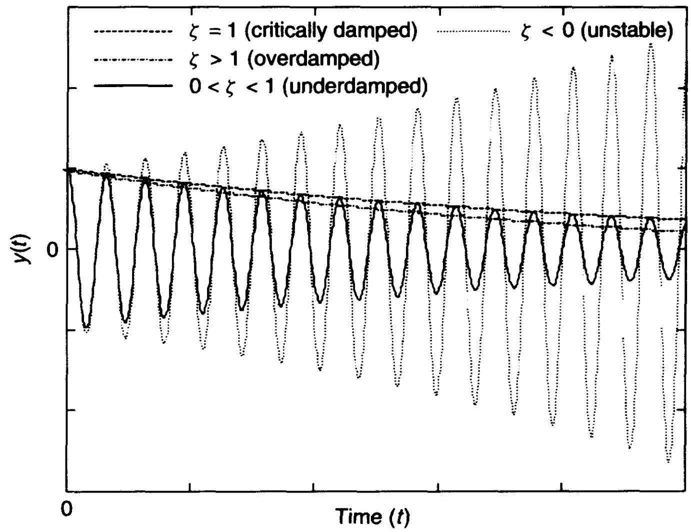


**Figure 2.24** Dependence of the response of a second order system on the damping-ratio, $\varsigma$

the response is oscillatory with amplitude decreasing with time (called the *underdamped case*), while for $\varsigma > 1$, the response decays exponentially (called the *overdamped case*). Clearly, the larger the value of the damping-ratio, $\varsigma$, the faster the response decays to zero. The case for which $\varsigma < 0$ denotes a response with *exponentially increasing amplitude*. A response, $y(t)$, whose limit as $t \to \infty$, either does not exist or is *infinite*, is called an *unbounded* response. Clearly, $\varsigma < 0$ case has an unbounded response. As soon as we see a linear system producing an unbounded response to a *bounded* input (i.e. an input whose finite limit exists as $t \to \infty$) and *finite* initial conditions, we call the system *unstable*. A further discussion of *stability* follows a little later.

Locations of poles and zeros in the Laplace domain determine the characteristics of a linear, time-invariant system. Some indication of the locations of a poles and zeros can be obtained from the frequency response, $G(i\omega)$. Let us go back to Figure 2.20, showing the Bode plots of the electrical system of Examples 2.8 and 2.9. Due to the presence of a zero at the origin (see Eq. (2.80)), there is a phase of $90°$ and a *non-zero* (dB) gain at $\omega = 0$. The presence of a complex conjugate pair of poles is indicated by a peak in the gain plot and a phase change of $180°$. The difference between the number of zeros and poles in a system affects the phase and the *slope* of the Bode gain plot with frequency (in units of dB per *decade* of frequency), when the frequency is very large (i.e. in the limit $\omega \to \infty$). From Eq. (2.79), we can say the following about gain-slope and phase in the high-frequency limit:

$$\lim_{\omega\to\infty} d\{20\log_{10}|G(i\omega)|\}/d\omega \approx 20(m-n) \text{ dB/decade}$$
$$\lim_{\omega\to\infty} \phi(\omega) \approx \begin{bmatrix} (m-n)90° & \text{if } K > 0 \\ (m-n)90° - 180° & \text{if } K < 0 \end{bmatrix} \tag{2.85}$$

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Note that the expressions in Eq. (2.85) are only *approximate*. For example, the transfer function in Eq. (2.80) has $K = 0.5$, $m = 1$, and $n = 2$, which implies that the gain-slope and phase in the limit $\omega \to \infty$ should be $-20$ dB/decade and $-90°$, respectively. These values are very good estimates (the phase is exactly $-90°$) of the frequency response plotted in Figure 2.20.

### *Example 2.10*

Consider a linear model describing the longitudinal dynamics of an aircraft (Figure 2.25). *Three* different output variables (in the Laplace domain) are of interest when the aircraft is displaced from the equilibrium point (defined by a constant *angle of attack*, $\alpha_0$, a constant longitudinal *velocity*, $v_0$, and a constant *pitch-angle*, $\theta_0$): the *change* in airspeed, $v(s)$, the *change* in the angle of attack, $\alpha(s)$, and the change in pitch angle, $\theta(s)$. The input variable in the Laplace domain is the *elevator angle*, $\delta(s)$. The three transfer functions separately defining the relationship between the input, $\delta(s)$, and the three respective outputs, $v(s)$, $\alpha(s)$, and $\theta(s)$, are as follows:

$$v(s)/\delta(s) = -0.0005(s-70)(s+0.5)/[(s^2+0.005s+0.006)(s^2+s+1.4)] \tag{2.86}$$

$$\alpha(s)/\delta(s) = -0.02(s+80)(s^2+0.0065s+0.006)/[(s^2+0.005s+0.006)(s^2+s+1.4)] \tag{2.87}$$

$$\theta(s)/\delta(s) = -1.4(s+0.02)(s+0.4)/[(s^2+0.005s+0.006)(s^2+s+1.4)] \tag{2.88}$$

It should be noted that all three transfer functions have the *same* denominator polynomial, $(s^2+0.005s+0.006)(s^2+s+1.4)$. Since we know that the denominator polynomial equated to zero denotes the characteristic equation of the system, we can write the characteristic equation for the aircraft's longitudinal dynamics as

$$(s^2+0.005s+0.006)(s^2+s+1.4) = 0 \tag{2.89}$$

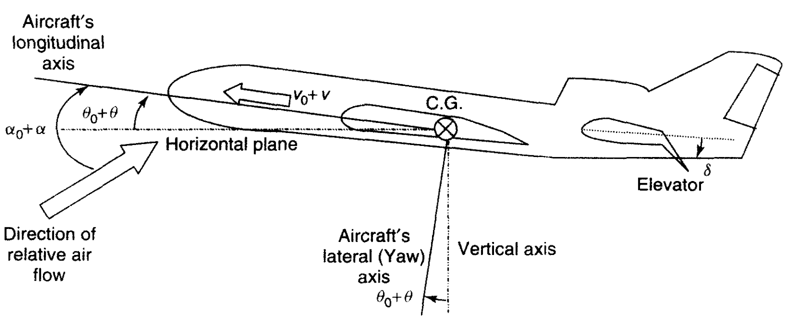


**Figure 2.25** Longitudinal dynamics of an airplane, with outputs $\alpha$, $\theta$, and $v$ denoting small changes in angle of attack, pitch angle, and velocity component along longitudinal axis, respectively, and input, elevator deflection, $\delta$. The equilibrium condition is denoted by $\alpha = \theta = v = \delta = 0$

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Equation (2.89) indicates that the systems complex poles are given by *two quadratic factors* $(s^2 + 0.005s + 0.006)$ and $(s^2 + s + 1.4)$. Comparing the result with that of Example 2.9, where the quadratic factor in the characteristic polynomial was expressed as $s^2 + 2\varsigma\omega_n s + \omega_n^2$, we can see that here we should expect *two* values of the natural frequency, $\omega_n$, and the damping-ratio, $\varsigma$, i.e. one set of values for each of the *two* quadratic factors. These values are the following:

(a) $\varsigma = 0.4226$; $\omega_n = 1.1832$ rad/s (*short-period mode*)

(b) $\varsigma = 0.0323$; $\omega_n = 0.0775$ rad/s (*long-period*, or *phugoid mode*)

Using MATLAB's Control System Toolbox (CST) command *damp*, the damping-ratio and natural frequency associated with each quadratic factor in the characteristic equation can be easily obtained as follows:

```
>>a=[1 0.005 0.006]; damp(a) % first quadratic factor <enter>

Eigenvalue                 Damping         Freq. (rad/sec)
-0.0025+0.0774i            0.0323          0.0775
-0.0025-0.0774i            0.0323          0.0775

>>b=[1 1 1.4]; damp(b) % second quadratic factor <enter>

Eigenvalue                 Damping         Freq. (rad/sec)
-0.5000+1.0724i            0.4226          1.1832
-0.5000-1.0724i            0.4226          1.1832
```

Note that the CST command *damp* also lists the *eigenvalues*, which are nothing but the roots of the characteristic polynomial (same as the *poles* of the system). We will discuss the *eigenvalues* in Chapter 3. (Alternatively, we could have used the intrinsic MATLAB function *roots* to get the pole locations as the roots of each quadratic factor.) As expected, the poles for each quadratic factor in the characteristic equation are complex conjugates. Instead of calculating the roots of each quadratic factor separately, we can multiply the two quadratic factors of Eq. (2.89) using the intrinsic MATLAB command *conv*, and then directly compute the roots of the characteristic polynomial as follows:

```
>>damp(conv(a,b))% roots of the characteristic polynomial <enter>

Eigenvalue                 Damping         Freq. (rad/sec)
-0.0025+0.0774i            0.0323          0.0775
-0.0025-0.0774i            0.0323          0.0775
-0.5000+1.0724i            0.4226          1.1832
-0.5000-1.0724i            0.4226          1.1832
```

The pair of natural frequencies and damping-ratios denote two *natural modes* of the system, i.e. the two ways in which one can excite the system. The first mode is highly damped, with a larger natural frequency (1.1832 rad/s), and is called the *short-period mode* (because the *time-period* of the oscillation, $T = 2\pi/\omega_n$ is

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smaller for this mode). The second characteristic mode is very lightly damped with a smaller natural frequency (0.0775 rad/s) – hence, a longer time-period – and is called the *long-period* (or *phugoid*) *mode*. While an arbitrary input will excite a response containing both of these modes, it is sometimes instructive to study the two modes separately. There are special elevator inputs, $\delta(s)$, which largely excite either one or the other mode at a time. (You may refer to Blakelock [3] for details of longitudinal dynamics and control of aircraft and missiles.)

We now examine the Bode plots of each of the *three* transfer functions, $v(s)/\delta(s)$, $\alpha(s)/\delta(s)$, and $\theta(s)/\delta(s)$, respectively, to see how much is each output variable influenced by each of the two characteristic modes. Figures 2.26, 2.27, and 2.28 show the gain and phase Bode plots for the three transfer functions in the limit $s = i\omega$ (they are the frequency responses of the concerned output variable). Using Control Systems Toolbox (CST), these plots are directly obtained by the command *bode*, after constructing each transfer function using the LTI object *tf*. Bode plot of transfer function $v(s)/\delta(s)$ (Figure 2.26) is generated using the following MATLAB statements:

```
>>a=[1 -70]; b=[1 0.5]; num=-0.0005*conv(a,b) <enter>

num =
 -0.0005  0.0348  0.0175

>>a=[1 0.005 0.006]; b=[1  1  1.4]; den=conv(a,b) <enter>

den =
  1.0000  1.0050  1.4110  0.0130  0.0084
```

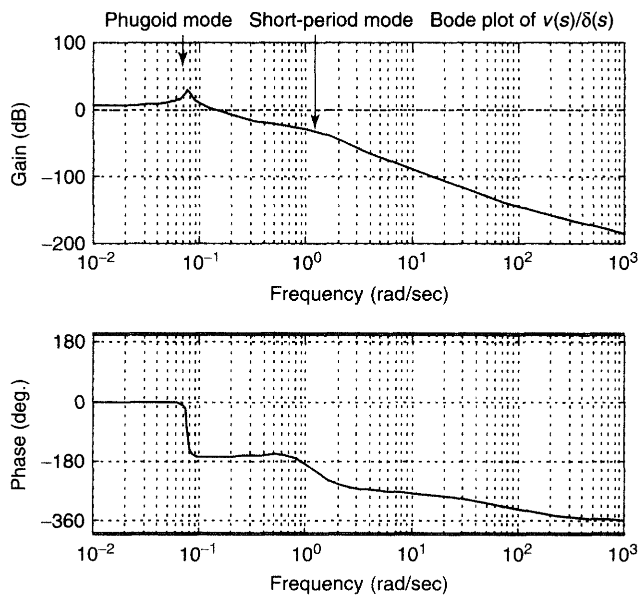


**Figure 2.26** Bode plot of the aircraft's transfer function $v(s)/\delta(s)$ with $s = i\omega$

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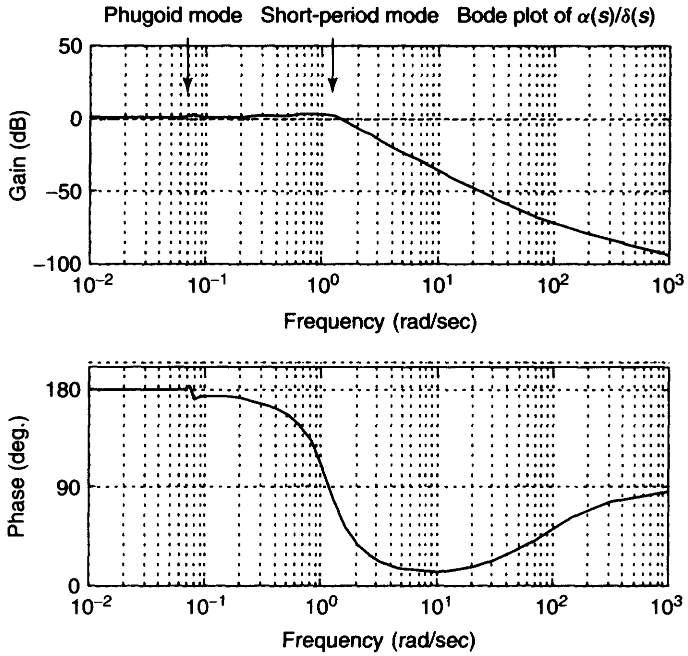


**Figure 2.27** Bode plot of the aircraft's transfer function $\alpha(s)/\delta(s)$ with $s = i\omega$

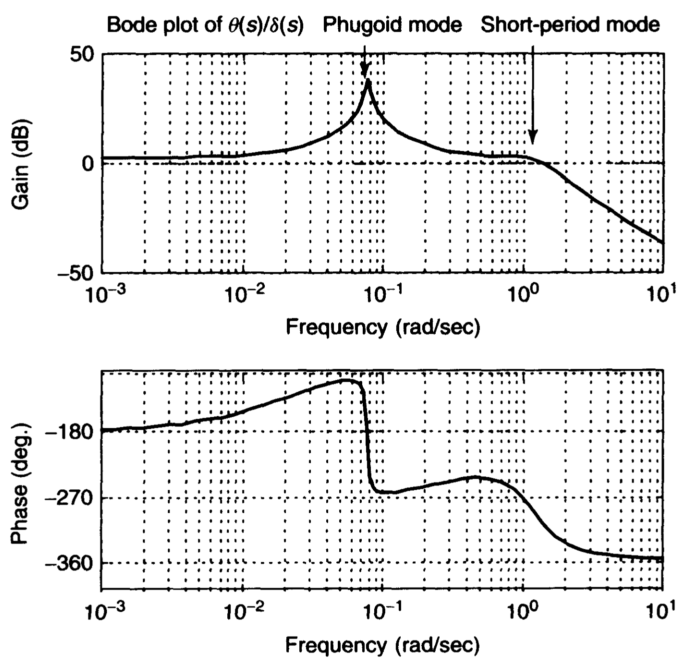


**Figure 2.28** Bode plot of the aircraft's transfer function $\theta(s)/\delta(s)$ with $s = i\omega$

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```
>>vdelta = tf(num,den) <enter>

Transfer function:
  -0.0005 s^2+0.03475 s+0.0175
------------------------------------------
s^4+1.005 s^3+1.411 s^2+0.013 s+0.0084

>>bode(vdelta) % Figure 2.26 <enter>
```

The Bode plot of transfer function $\alpha(s)/\delta(s)$ (Figure 2.27) is generated using the following MATLAB statements:

```
>>a=[1 80];b=[1  0.0065  0.006]; num=-0.02*conv(a,b) <enter>

num =
 -0.0200  -1.6001  -0.0105  -0.0096

>>alphadelta = tf(num,den) <enter>

Transfer function:
  -0.02 s^3-1.6 s^2-0.01052 s-0.0096
------------------------------------------
s^4+1.005 s^3+1.411 s^2+0.013 s+0.0084

>>bode(alphadelta) % Figure 2.27 <enter>
```

(Note that the denominator polynomial, *den*, of $\alpha(s)/\delta(s)$ is same as that of $v(s)/\delta(s)$, and does not have to be re-calculated.)

Finally, the Bode plot of transfer function $\theta(s)/\delta(s)$ (Figure 2.28) is generated using the following MATLAB statements:

```
>>a=[1  0.02]; b=[1  0.4]; num=-1.4*conv(a,b) <enter>

num =
  -1.4000  -0.5880  -0.0112

>>thetadelta = tf(num,den) <enter>

Transfer function:
  -1.4 s^2-0.588 s-0.0112
----------------------------------------------
s^4+1.005 s^3+1.411 s^2+0.013 s+0.0084

>>bode(thetadelta) % Figure 2.28 <enter>
```

From the Bode plots (Figures 2.26–2.28), we can note the natural frequencies of the phugoid and the short period modes, respectively, as either the *peaks* or *changes of slope* (called *breaks*) in the respective *gain* plots. The peaks due to complex poles sometimes disappear due to the presence of zeros in the vicinity of the poles. As expected, the natural frequencies agree with the values already

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calculated from the characteristic polynomial, because all the three transfer functions have the same characteristic (denominator) polynomial. Figure 2.26 shows that the magnitude (gain) of $v(i\omega)/\delta(i\omega)$ at the short period natural frequency is very small, which indicates that the short period mode oscillation is characterized by very small changes in forward velocity, $v(i\omega)$, which can be neglected (i.e. $v(i\omega) \approx 0$) to obtain a *short period approximation*. As expected, near each natural frequency the phase changes by 180°, except for the phugoid mode in $\alpha(i\omega)/\delta(i\omega)$ (Figure 2.27). The latter strange behavior of the phugoid mode is due to the fact that in the transfer function $\alpha(i\omega)/\delta(i\omega)$, one of the numerator quadratics (i.e. a pair of complex zeros) *almost cancels out* the quadratic corresponding to the phugoid mode in the denominator polynomial (i.e. a pair of complex poles), indicating that there is *essentially* no change in the angle-of-attack, $\alpha(i\omega)$, in the phugoid mode. Also, the magnitude (gain) of $\alpha(i\omega)/\delta(i\omega)$ at the phugoid natural frequency is seen to be very small in Figure 2.27 as compared to the gain at the same frequency in Figures 2.26 and 2.28. The fact that the phugoid oscillation does not involve an appreciable change in the angle-of-attack, $\alpha(i\omega)$, forms the basis of the *phugoid approximation* in which $\alpha(i\omega) \approx 0$. However, Figure 2.28 shows that considerable magnitude (gain) of $\theta(i\omega)/\delta(i\omega)$ exists at *both* short period and the phugoid natural frequencies. Hence, both modes essentially consist of oscillations in the pitch angle, $\theta(i\omega)$. The present example shows how one can obtain an insight into a system's behavior just by analyzing the frequency response of its transfer function(s).

Note from Figures 2.26–2.28 that the gains of all three transfer functions decay rapidly with frequency at high frequencies. Such a decay in the gain at high frequencies is a desirable feature, called *roll-off*, and provides *attenuation* of high frequency *noise* arising due to *unmodeled* dynamics in the system. We will define *sensitivity* (or *robustness*) of a system to transfer function variations later in this chapter, and formally study the effects of *noise* in Chapter 7. Using Eq. (2.85), we can estimate the high-frequency gain-slope and phase of the three transfer functions given by Eqs. (2.86)–(2.88). For $v(s)/\delta(s)$, $K < 0$, $m = 2$, and $n = 4$, which implies a gain-slope (or roll-off) of $-40$ dB/decade and a phase of $-360°$ (or 0°) in the limit $\omega \to \infty$, which are confirmed in Figure 2.26. For $\alpha(s)/\delta(s)$, $K < 0$, $m = 3$, and $n = 4$, which implies a roll-off of $-20$ dB/decade and a phase of $-270°$ (or 90°) in the limit $\omega \to \infty$, which are evident in Figure 2.27. Finally, for $\theta(s)/\delta(s)$, $K < 0$, $m = 2$, and $n = 4$, which implies a gain-slope (or roll-off) of $-40$ dB/decade and a phase of $-360°$ (or 0°) in the limit $\omega \to \infty$, which are also seen in Figure 2.28.

The transfer function $v(s)/\delta(s)$ has a peculiarity which is absent in the other two transfer functions – namely, a *zero* at $s = 70$. A system with transfer function having *poles* or *zeros* in the *right-half* $s$-plane is called a *non-minimum phase* system, while a system with all the poles and zeros in the left-half $s$-plane, or on the imaginary axis is called a *minimum phase* system. We will see below that systems which have *poles* in the right-half $s$-plane are *unstable*. Hence, *stable* non-minimum phase systems have only zeros in the right-half $s$-plane, such as the system denoted by $v(s)/\delta(s)$. Stable non-minimum phase systems have a markedly different phase in the limit $\omega \to \infty$ (we may have to add or subtract 360° to find non-minimum phase

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from Eq. (2.85)), when compared to a *corresponding* minimum phase system (i.e. a similar system with no zeros in the right-half $s$-plane). This usually results in an unacceptable transient response. A non-minimum phase system with only one right-half plane zero (such as $v(s)/\delta(s)$) results in a transient response which is of *opposite sign* when compared to the input. Popular examples of such systems are aircraft or missiles controlled by forces applied *aft* of the center of mass. For this reason, a right-half plane zero in an aircraft (or missile) transfer function is called 'tail-wags-the-dog zero'. Control of non-minimum phase systems requires special attention.

Before we can apply the transfer function approach to a general system, we must know how to derive Laplace transform (and inverse Laplace transform) of some frequently encountered functions. This information is tabulated in Table 2.1, using the definitions and properties of the Laplace transform (Eqs. (2.59)–(2.70)). Note that Table 2.1 gives the Laplace transform of some commonly encountered functions, $f(t)$, which are defined for $t \geq 0$. At $t = 0$, $f(t)$ can have a discontinuity, such as $f(t) = u_s(t)$ or $f(t) = \delta(t)$. It is interesting to see in Table 2.1 that the Laplace transform of the unit impulse function, $\delta(t)$, is unity, while that of the unit step function, $u_s(t)$, is $1/s$. Since $du_s(t)/dt = \delta(t)$, the Laplace transforms of these two singularity functions agree with the properties given by Eqs. (2.66) and (2.68).

**Table 2.1** Laplace transforms of some common functions

| S. No. | $f(t)$ $(t \geq 0)$ | $F(s) = \mathcal{L}f(t) = \int_0^\infty e^{-st} f(t)\,dt$ |
|---|---|---|
| 1 | $e^{-at}$ | $1/(s+a)$ |
| 2 | $e^{-at}f(t)$ | $F(s+a)$ |
| 3 | $t^n$ | $n!/s^{n+1}$ |
| 4 | Unit Step Function, $u_s(t)$ | $1/s$ |
| 5 | $\sin(\omega t)$ | $\omega/(s^2+\omega^2)$ |
| 6 | $\cos(\omega t)$ | $s/(s^2+\omega^2)$ |
| 7 | $f^{(k)}(t)$ | $s^kF(s) - s^{k-1}f(0) - s^{k-2}f^{(1)}(0) - \cdots - f^{(k-1)}(0)$ |
| 8 | $\int_{-\infty}^{t} f(t)\,dt$ | $F(s)/s + (1/s)\int_{-\infty}^{0} f(t)\,dt$ |
| 9 | Unit Impulse Function, $\delta(t)$ | 1 |

### *Example 2.11*

Consider a system with the following transfer function:

$$G(s) = (s+3)/[(s+1)(s+2)] \tag{2.90}$$

The second order system (denominator polynomial is of degree 2) has a zero, $z_1 = -3$, and two poles, $p_1 = -1$ and $p_2 = -2$. Let us assume that the system has the

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input, $u(t)$, and initial conditions as follows:

$$u(t) = 0, y(0) = y_0, y^{(1)}(0) = 0 \tag{2.91}$$

Since $G(s) = Y(s)/U(s)$ when the *initial conditions are zero* (which is not the case here), we cannot directly use the transfer function to determine the system's response, $y(t)$, for $t > 0$. Let us first derive the system's governing differential equation by applying inverse Laplace transform to the transfer function (with zero initial conditions, because that is how a transfer function is defined) as follows:

$$(s+1)(s+2)Y(s) = (s+3)U(s) \tag{2.92}$$

or

$$s^2Y(s) + 3sY(s) + 2Y(s) = sU(s) + 3U(s) \tag{2.93}$$

and

$$\mathcal{L}^{-1}[s^2Y(s) + 3sY(s) + 2Y(s)] = \mathcal{L}^{-1}[sU(s) + 3U(s)] \tag{2.94}$$

which, using the real differentiation property (Eq. (2.67)) with zero initial conditions for both input, $u(t)$, and output, $y(t)$, yields the following differential equation:

$$y^{(2)}(t) + 3y^{(1)}(t) + 2y(t) = u^{(1)}(t) + 3u(t) \tag{2.95}$$

Now, we can apply the Laplace transform to this governing differential equation using real differentiation property *with the input and the initial conditions* given by Eq. (2.91) as

$$\mathcal{L}[y^{(2)}(t) + 3y^{(1)}(t) + 2y(t)] = \mathcal{L}[u^{(1)}(t) + 3u(t)] \tag{2.96}$$

or

$$s^2Y(s) - sy_0 + 3sY(s) - 3y_0 + 2Y(s) = 0 \tag{2.97}$$

and it follows that

$$Y(s) = (s+3)y_0/[(s+1)(s+2)] \tag{2.98}$$

We can express $Y(s)$ as

$$Y(s) = y_0[2/(s+1) - 1/(s+2)] \tag{2.99}$$

Equation (2.99) is called the *partial fraction expansion* of Eq. (2.98), where the contribution of each pole is expressed separately as a fraction and added up. In Eq. (2.99) the two numerator coefficients, 2 and $-1$, corresponding to the two fractions are called the *residues*.

The output, $y(t)$, of the system can then be obtained by applying inverse Laplace transform to Eq. (2.99) for $t \geq 0$ as

$$y(t) = y_0\{\mathcal{L}^{-1}[2/(s+1)] + \mathcal{L}^{-1}[-1/(s+2)]\} \tag{2.100}$$

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