Note that the time-step for calculating the impulse response is *smaller* than inverse of the *largest natural frequency* of the system (1.18 rad/s). Similarly, the step response, $s(t)$, is computed using *stepresp* and plotted as follows:

```
>>[s,t] = stepresp(num,den,0,0.5,250); plot(t,s) <enter>
```

The resulting plots of impulse and step responses are shown in Figure 2.29. Note that the plot of the impulse response clearly shows an initial, well damped oscillation of high-frequency (short-period mode) and a *lightly damped*, long-period oscillation (phugoid mode). This behavior meets our expectation from the natural frequencies and damping-ratios of the two modes calculated in Example 2.10 (recall that the natural frequency and damping of the short-period mode are more than 10 times those of the phugoid mode). It is also clear that the impulse response excites both the modes, while the step response is dominated by the phugoid mode. The time taken by the phugoid mode to decay to zero is an indicator of the *sluggishness* of the longitudinal dynamics of the airplane. The non-minimum phase character of the transfer function due to the zero at $s = 70$ is evident in the large initial undershoot in the impulse response.

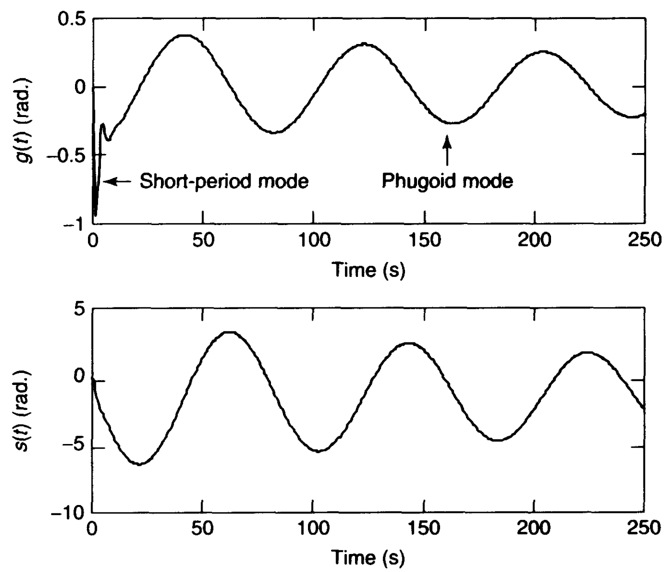

Figure 2.29 Impulse response, $g(t)$, and step response, $s(t)$, for the aircraft transfer function, $\theta(s)/\delta(s)$ (Example 2.13)

2.6 Response to Arbitrary Inputs

After learning how to find step and impulse responses of a given linear system, the next logical step is to find the response to an arbitrary input, $u(t)$, which is applied to a linear system with zero initial conditions. At the end of Section 2.2 (Figure 2.16), we saw how an arbitrary function can be expressed as a sum (or integral) of the suitably scaled impulse functions, $\delta(t-a)$, as given by Eq. (2.36). Since we know how to find the response of a linear, time-invariant system to a unit impulse input for zero initial conditions, we can apply the superposition principle to get the linear system's response, $y(t)$, to an arbitrary input, $u(t)$, which is expressed as

$$u(t) = \sum_{\tau=-\infty}^{\infty} u(\tau)\Delta\tau\delta(t-\tau) \tag{2.117}$$

and the response to the impulse input $u(\tau)\Delta\tau\delta(t-\tau)$ applied at $t=\tau$ is given by

$$\Delta y(t,\tau) = u(\tau)\Delta\tau g(t-\tau) \tag{2.118}$$

where $g(t-\tau)$ is the impulse response to unit impulse, $\delta(t-\tau)$, applied at $t=\tau$. Then by the superposition principle (Eq. (2.36)), the response to a linear combination of impulses (given by Eq. (2.35)) is nothing else but a linear combination of individual impulse responses, $g(t-\tau)$, each multiplied by the corresponding factor, $u(\tau)\Delta\tau$. Hence, we can write the response, $y(t)$, as

$$y(t) = \sum_{\tau=-\infty}^{\infty} u(\tau)\Delta\tau g(t-\tau) \tag{2.119}$$

or, in the limit $\Delta\tau \to 0$,

$$y(t) = \int_{-\infty}^{\infty} u(\tau)g(t-\tau)\,d\tau \tag{2.120}$$

Equation (2.120) is one of the most important equations in control theory, since it lets us determine a linear system's response to an arbitrary input. The integral on the right-hand side of Eq. (2.120) is called the *superposition integral* (or *convolution integral*). Note that we can apply a change of integration variable, and show that the convolution integral is *symmetric* with respect to $u(t)$ and $g(t)$ as follows:

$$y(t) = \int_{-\infty}^{\infty} u(\tau)g(t-\tau)\,d\tau = \int_{-\infty}^{\infty} u(t-\tau)g(\tau)\,d\tau \tag{2.121}$$

Most commonly, the input, $u(t)$, is non-zero only for $t>0$. Also, since $g(t-\tau)=0$ for $\tau>t$, we can change the upper limit of the convolution integral to t. Hence, the convolution integral becomes the following:

$$y(t) = \int_{0}^{t} u(\tau)g(t-\tau)\,d\tau = \int_{0}^{t} u(t-\tau)g(\tau)\,d\tau \tag{2.122}$$

We could have obtained Eq. (2.122) alternatively by applying inverse Laplace transform to $Y(s)=G(s)U(s)$ (since $G(s)$ is the Laplace transform of $g(t)$).

Example 2.14

We can use the convolution integral to obtain the step response, $s(t)$, of the second order system of Example 2.12 as follows:

$$s(t) = \int_0^t u_s(t-\tau)g(\tau)\,d\tau \tag{2.123}$$

where $u_s(t)$ is the unit step function and the impulse response, $g(t)$, is given by Eq. (2.110). Substituting Eq. (2.110) into Eq. (2.123), and noting that $u_s(t-\tau) = 1$ for $\tau < t$ and $u_s(t-\tau) = 0$ for $\tau > t$, we get the following expression for the step response:

$$s(t) = \int_0^t \mathrm{e}^{-15\tau}[0.5\cos(999.9\tau) - 0.0075\sin(999.9\tau)]\,d\tau \tag{2.124}$$

Carrying out the integration in Eq. (2.124) by parts, we get the same expression for $s(t)$ as we obtained in Eq. (2.114), i.e. $s(t) = 5\times 10^{-4}\,\mathrm{e}^{-15t}\sin(999.9t)$ for $t \geq 0$.

The use of convolution integral of Eq. (2.120) can get difficult to evaluate by hand if the input $u(t)$ is a more complicated function than the unit step function. Hence, for a really arbitrary input, it is advisable to use a *numerical approximation* of the convolution integral as a *summation* over a large number of finite time-steps, $\Delta\tau$. Such an

Table 2.4 Listing of the M-file *response.m*, which calculates the response of a strictly proper, single-input, single-output system to an arbitrary input

response.m

```
function y=response(num,den,t,u);
%Program for calculation of the response of strictly proper SISO systems
%to arbitrary input by the convolution integral.
%num = numerator polynomial coefficients of transfer function
%den = denominator polynomial coefficients of transfer function
%(Coefficients of 'num' and 'den' are specified as a row vector, in
%decreasing powers of 's')
%t = row vector of time points (specified by the user)
%u = vector of input values at the time points contained in t.
%y = calculated response
%copyright(c)2000 by Ashish Tewari
%
%Calculate the time-step:-
dt=t(2)-t(1);
m=size(t,2)
tf=t(m);
%Calculate the convolution integral:-
y=zeros(size(t));
G=y;
[g,T]=impresp(num,den,t(1),dt,tf);
for i=1:m
y=y+dt*u(i)*[G(1:i-1) g(1:m-i+1)];
end
```

approximation of an integral by a summation is called *quadrature*, and there are many numerical techniques available of varying efficiency for carrying out quadrature. The simplest numerical integration (or quadrature) is the assumption that the integrand is *constant* in each time interval, $\Delta\tau$, which is the same as Eq. (2.119). Hence, we can use Eq. (2.119) with a *finite* number of time steps, $\Delta\tau$, to evaluate the convolution integral, and the response to an arbitrary input, $u(t)$. Such a numerical evaluation of the convolution integral is performed by the M-file, called *response.m*, listed in Table 2.4. Note that *response* calls *impresp.m* internally to evaluate the impulse response, $g(t-\tau)$, for each time interval, $\Delta\tau$. The M-file *response.m* is called as follows:

```
>>y = response(num,den,t,u) <enter>
```

where *num*, *den* are the numerator and denominator polynomial coefficients of the transfer function, $G(s)$, (same as in *impresp* and *stepresp*), t is the row vector containing time points at which the response, y, is desired, and u is a vector containing values of the applied arbitrary input at the time points contained in t.

Example 2.15

Let us determine the pitch response, $\theta(t)$, of the aircraft transfer function, $\theta(s)/\delta(s)$, (Examples 2.10, 2.13) if the applied input is $\delta(t) = 0.01t.\sin(1.3t)$, beginning at $t = 0$. We first specify the time vector, t, and input vector, u, as follows:

```
>>t=0:0.1:20; u=0.01*t.*sin(1.3*t); <enter>
```

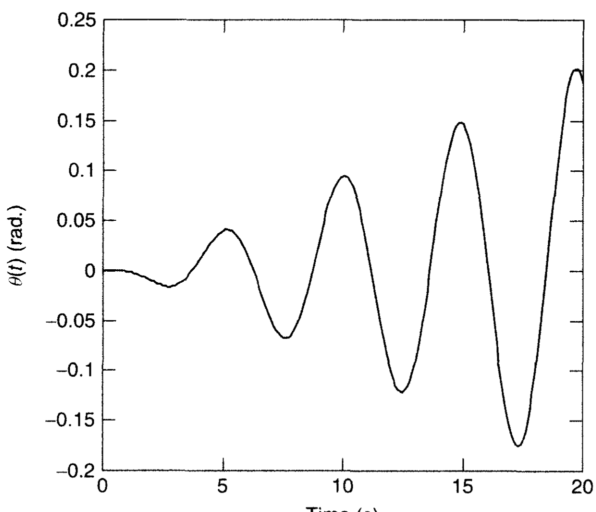

Figure 2.30 Response of the pitch angle, $\theta(t)$, for the aircraft when the elevator deflection is $\delta(t) = 0.01t.\sin(1.3t)$ radians