3

State-Space Representation

3.1 The State-Space: Why Do I Need It?

In Chapter 1, we defined the *state* of a system as any set of quantities which must be specified at a given time in order to completely determine the behavior of the system. The quantities constituting the state are called the *state variables*, and the hypothetical space spanned by the state variables is called the *state-space*. In a manner of speaking, we put the cart before the horse – we went ahead and defined the state before really understanding what it was. In the good old car-driver example, we said that the state variables could be the car's speed and the positions of all other vehicles on the road. We also said that the state variables are not unique; we might as well have taken the velocities of all other vehicles relative to the car, and the position of the car with respect to the road divider to be the state variables of the car-driver system. Let us try to understand what the state of a system really means by considering the example of a simple pendulum.

Example 3.1

Recall from Example 2.2 that the governing differential equation for the motion of a simple pendulum on which no external input is applied (Figure 2.3) is given by Eq. (2.8). If we apply a torque, $M(t)$, about the hinge, O, as an input to the pendulum, the governing differential equation can be written as

$$L\theta^{(2)}(t) + g\sin(\theta(t)) = M(t)/(\mathrm{mL}) \tag{3.1}$$

where $\theta^{(2)}(t)$ represents the second order time derivative of $\theta(t)$, as per our notation (i.e. $d^2\theta(t)/dt^2 = \theta^{(2)}(t)$). Let the output of the system be the angle, $\theta(t)$, of the pendulum. We would like to determine the state of this system. To begin, we must know how many quantities (i.e. state variables) need to be specified to completely determine the motion of the pendulum. Going back to Chapter 2, we know that for a system of order n, we have to specify precisely n *initial conditions* to solve the governing differential equation. Hence, it must follow that the *state* of an nth order system should consist of *precisely* n state variables, which must be specified at some time (e.g. $t = 0$) as initial conditions in order to completely determine the solution to the governing differential equation. Here we are dealing with a *second order* system – which implies that the state must consist of *two* state variables. Let

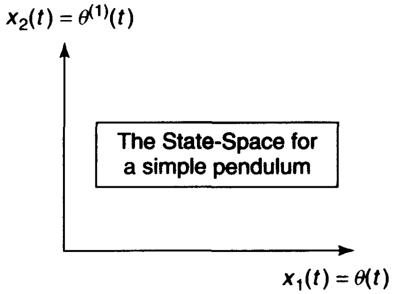

Figure 3.1 The two-dimensional state-space for a simple pendulum (Example 3.1)

us call these state variables $x_1(t)$ and $x_2(t)$, and arbitrarily choose them to be the following:

$$x_1(t) = \theta(t) \tag{3.2}$$

$$x_2(t) = \theta^{(1)}(t) \tag{3.3}$$

The *state-space* is thus *two-dimensional* for the simple pendulum whose axes are $x_1(t)$ and $x_2(t)$ (Figure 3.1).

It is now required that we express the governing differential equation (Eq. (3.1)) in terms of the state variables defined by Eqs. (3.2) and (3.3). Substituting Eqs. (3.2) and (3.3) into Eq. (3.1), we get the following first order differential equation:

$$x_2^{(1)}(t) = M(t)/(mL^2) - (g/L)\sin(x_1(t)) \tag{3.4}$$

Have we *transformed* a second order differential equation (Eq. (3.1)) into a first order differential equation (Eq. (3.4)) by using the state-variables? *Not really*, because there is a *another* first order differential equation that we have forgotten about – the one obtained by substituting Eq. (3.2) into Eq. (3.3), and written as

$$x_1^{(1)}(t) = x_2(t) \tag{3.5}$$

Equations (3.4) and (3.5) are *two first order* differential equations, called the *state equations*, into which the governing equation (Eq. (3.1)) has been transformed. The *order* of the system, which is its important characteristic, remains *unchanged* when we express it in terms of the state variables. In addition to the state equations (Eqs. (3.4) and (3.5)), we need an *output equation* which defines the relationship between the output, $\theta(t)$, and the state variables $x_1(t)$ and $x_2(t)$. Equation (3.2) simply gives the output equation as

$$\theta(t) = x_1(t) \tag{3.6}$$

The state equations, Eqs. (3.4) and (3.5), along with the output equation, Eq. (3.6), are called the *state-space representation* of the system.

Instead of choosing the state variables as $\theta(t)$ and $\theta^{(1)}(t)$, we could have selected a different set of state variables, such as

$$x_1(t) = L\theta(t) \tag{3.7}$$

and

$$x_2(t) = L^2\theta^{(1)}(t) \tag{3.8}$$

which would result in the following state equations:

$$x_1^{(1)}(t) = x_2(t)/L \tag{3.9}$$

$$x_2^{(1)}(t) = M(t)/m - gL\sin(x_1(t)/L) \tag{3.10}$$

and the output equation would be given by

$$\theta(t) = x_1(t)/L \tag{3.11}$$

Although the state-space representation given by Eqs. (3.9)–(3.11) is different from that given by Eqs. (3.4)–(3.6), both descriptions are for the same system. Hence, we expect that the solution of *either* set of equations would yield the *same* essential characteristics of the system, such as performance, stability, and robustness. Hence, the state-space representation of a system is not *unique*, and all *legitimate* state-space representations should give the same system characteristics. What do we mean by a *legitimate* state-space representation? While we have freedom to choose our state variables, we have to ensure that we have chosen the *minimum* number of state variables that are required to describe the system. In other words, we should not have *too many* or *too few* state variables. One way of ensuring this is by taking *precisely* n state variables, where n is the order of the system. If we are deriving state-space representation from the system's governing differential equation (such as in Example 3.1), the number of state-variables is easily determined by the order of the differential equation. However, if we are deriving the state-space representation from a transfer function (or transfer matrix), some poles may be *canceled* by the zeros, thereby yielding an *erroneous* order of the system which is *less* than the correct order.

Example 3.2

Consider a system with input, $u(t)$, and output, $y(t)$, described by the following differential equation:

$$y^{(2)}(t) + (b-a)y^{(1)}(t) - ab\ y(t) = u^{(1)}(t) - au(t) \tag{3.12}$$

where a and b are positive constants. The transfer function, $Y(s)/U(s)$, of this system can be obtained by taking the Laplace transform of Eq. (3.12) with *zero initial conditions*, and written as follows:

$$Y(s)/U(s) = (s-a)/[s^2 + (b-a)s - ab] = (s-a)/[(s-a)(s+b)] \tag{3.13}$$

In Eq. (3.13), if we cannot resist the temptation to cancel the pole at $s = a$ with the zero at $s = a$, we will be left with the following transfer function:

$$Y(s)/U(s) = 1/(s + b) \tag{3.14}$$

which yields the following *incorrect* differential equation for the system:

$$y^{(1)}(t) + by(t) = u(t) \tag{3.15}$$

Since the pole cancelled at $s = a$ has a *positive real part*, the actual system given by the transfer function of Eq. (3.13) is *unstable*, while that given by Eq. (3.14) is *stable*. Needless to say, basing a state-space representation on the transfer function given by Eq. (3.14) will be incorrect. This example illustrates one of the hazards associated with the transfer function description of a system, which can be avoided if we directly obtain state-space representation from the governing differential equation.

Another cause of illegitimacy in a state-space representation is when two (or more) state variables are *linearly dependent*. For example, if $x_1(t) = \theta(t)$ is a state variable, then $x_2(t) = L\theta(t)$ *cannot* be another state variable in the *same state-space representation*, because that would make $x_1(t)$ and $x_2(t)$ linearly dependent. You can demonstrate that with such a choice of state variables in Example 3.1, the state equations will *not* be *two* first order differential equations. In general, for a system of order n, if $x_1(t), x_2(t), \ldots, x_{n-1}(t)$ are state variables, then $x_n(t)$ is *not* a legitimate state variable if it can be expressed as a linear combination of the other state variables given by

$$x_n(t) = c_1x_1(t) + c_2x_2(t) + \cdots + c_{n-1}x_{n-1}(t) \tag{3.16}$$

where $c_1, c_2, \ldots, c_{n-1}$ are constants. Thus, while we have an unlimited choice in selecting state variables for a given system, we should ensure that their number is *equal* to the order of the system, and also that *each* state variable is *linearly independent* of the other state variables in a state-space representation.

In Chapter 2, we saw how *single-input*, *single-output*, *linear systems* can be designed and analyzed using the *classical* methods of frequency response and transfer function. The transfer function – or frequency response – representations of linear systems were indispensable before the wide availability of fast digital computers, necessitating the use of tables (such as the *Routh* table [1]) and graphical methods, such as *Bode*, *Nyquist*, *root-locus*, and *Nichols* plots for the analysis and design of control systems. As we saw in Chapter 2, the classical methods require a lot of complex variable analysis, such as interpretation of gain and phase plots and complex mapping, which becomes complicated for multivariable systems. Obtaining information about a multivariable system's time-response to an arbitrary input using classical methods is a difficult and indirect process, requiring inverse Laplace transformation. Clearly, design and analysis of modern control systems which are usually multivariable (such as Example 2.10) will be very difficult using the classical methods of Chapter 2.

In contrast to classical methods, the state-space methods work directly with the governing differential equations of the system in the time-domain. Representing the governing differential equations by first order state equations makes it possible to directly solve the state equations in time, using standard numerical methods and efficient algorithms on today's fast digital computers. Since the state equations are always of first order irrespective of the system's order or the number of inputs and outputs, the greatest advantage of state-space methods is that they *do not formally distinguish* between single-input, single-output systems and multivariable systems, allowing efficient design and analysis of multivariable systems with the same ease as for single variable systems. Furthermore, using state-space methods it is possible to directly design and analyze *nonlinear* systems (such as Example 3.1), which is utterly impossible using classical methods. When dealing with linear systems, state-space methods result in repetitive *linear algebraic* manipulations (such as matrix multiplication, inversion, solution of a linear matrix equation, etc.), which are easily programmed on a digital computer. This saves a lot of drudgery that is common when working with inverse Laplace transforms of transfer matrices. With the use of a high-level programming language, such as MATLAB, the linear algebraic manipulations for state-space methods are a breeze. Let us find a state-space representation for a multivariable nonlinear system.

Example 3.3

Consider an inverted pendulum on a moving cart (see Exercise 2.1), for which the governing differential equations are the following:

$$(M+m)x^{(2)}(t) + mL\theta^{(2)}(t)\cos(\theta(t)) - mL[\theta^{(1)}(t)]^2\sin(\theta(t)) = f(t) \tag{3.17}$$

$$mx^{(2)}(t)\cos(\theta(t)) + mL\theta^{(2)}(t) - mg\sin(\theta(t)) = 0 \tag{3.18}$$

where m and L are the mass and length, respectively, of the inverted pendulum, M is the mass of the cart, $\theta(t)$ is the angular position of the pendulum from the vertical, $x(t)$ is the horizontal displacement of the cart, $f(t)$ is the applied force on the cart in the same direction as $x(t)$ (see Figure 2.59), and g is the acceleration due to gravity. Assuming $f(t)$ to be the input to the system, and $x(t)$ and $\theta(t)$ to be the two outputs, let us derive a state-space representation of the system.

The system is described by *two* second order differential equations; hence, the order of the system is *four*. Thus, we need precisely four linearly independent state-variables to describe the system. When dealing with a physical system, it is often desirable to select *physical quantities* as state variables. Let us take the state variables to be the angular position of the pendulum, $\theta(t)$, the cart displacement, $x(t)$, the angular velocity of the pendulum, $\theta^{(1)}(t)$, and the cart's velocity, $x^{(1)}(t)$. We can arbitrarily number the state variables as follows:

$$x_1(t) = \theta(t) \tag{3.19}$$

$$x_2(t) = x(t) \tag{3.20}$$

$$x_3(t) = \theta^{(1)}(t) \tag{3.21}$$

$$x_4(t) = x^{(1)}(t) \tag{3.22}$$

From Eqs. (3.19) and (3.21), we get our first state-equation as follows:

$$x_1^{(1)}(t) = x_3(t) \tag{3.23}$$

while the second state-equation follows from Eqs. (3.20) and (3.22) as

$$x_2^{(1)}(t) = x_4(t) \tag{3.24}$$

The two remaining state-equations are derived by substituting Eqs. (3.19)–(3.22) into Eq. (3.17) and Eq. (3.18), respectively, yielding

$$x_3^{(1)}(t) = g\sin(x_1(t))/L - x_4^{(1)}(t)\cos(x_1(t))/L \tag{3.25}$$

$$\begin{aligned} x_4^{(1)}(t) &= [mL/(M+m)][x_3^{(1)}(t)]^2\sin(x_1(t)) \\ &\quad - [mL/(M+m)]x_3^{(1)}(t)\cos(x_1(t)) + f(t)/(M+m) \end{aligned} \tag{3.26}$$

The two output equations are given by

$$\theta(t) = x_1(t) \tag{3.27}$$

$$x(t) = x_2(t) \tag{3.28}$$

Note that due to the nonlinear nature of the system, we cannot express the last two state-equations (Eqs. (3.25), (3.26)) in a form such that each equation contains the *time derivative* of only *one* state variable. Such a form is called an *explicit form* of the state-equations. If the motion of the pendulum is *small* about the *equilibrium point*, $\theta = 0$, we can *linearize* Eqs. (3.25) and (3.26) by assuming $\cos(\theta(t)) = \cos(x_1(t)) = 1$, $\sin(\theta(t)) = \sin(x_1(t)) = x_1(t)$, and $[\theta^{(1)}(t)]^2\sin(\theta(t)) = [x_3^{(1)}(t)]^2 \sin(x_1(t)) = 0$. The corresponding *linearized* state equations can then be written in explicit form as follows:

$$x_3^{(1)}(t) = [(M+m)g/(ML)]x_1(t) - f(t)/(ML) \tag{3.29}$$

$$x_4^{(1)}(t) = -(mg/M)x_1(t) + f(t)/M \tag{3.30}$$

The *linearized* state-equations of the system, Eqs. (3.23), (3.24), (3.29), and (3.30), can be expressed in the following *matrix* form, where all coefficients are collected together by suitable *coefficient matrices*:

$$\begin{bmatrix} x_1^{(1)}(t) \\ x_2^{(1)}(t) \\ x_3^{(1)}(t) \\ x_4^{(1)}(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ (M+m)g/(ML) & 0 & 0 & 0 \\ -mg/M & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1/(ML) \\ 1/M \end{bmatrix} f(t) \tag{3.31}$$

with the output matrix equation given by

$$\begin{bmatrix} \theta(t) \\ x(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} f(t) \tag{3.32}$$

Note that the state-space representation of the linearized system consists of linear state-equations, Eq. (3.31), and a linear output equation, Eq. (3.32).

Taking a cue from Example 3.3, we can write the state-equations of a general linear system of order n, with m inputs and p outputs, in the following matrix form:

$$\mathbf{x}^{(1)}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{3.33}$$

and the general output equation is

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \tag{3.34}$$

where $\mathbf{x}(t) = [x_1(t); x_2(t); \ldots; x_n(t)]^T$ is the *state vector* consisting of n state variables as its elements, $\mathbf{x}^{(1)}(t) = [x_1^{(1)}(t); x_2^{(1)}(t); \ldots; x_n^{(1)}(t)]^T$ is the *time derivative of the state vector*, $\mathbf{u}(t) = [u_1(t); u_2(t); \ldots; u_r(t)]^T$ is the *input vector* consisting of r inputs as its elements, $\mathbf{y}(t) = [y_1(t); y_2(t); \ldots; y_p(t)]^T$ is the *output vector* consisting of p outputs as its elements, and $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ are the *coefficient matrices*. Note that the row dimension (i.e. the number of rows) of the state vector is equal to the order of the system, n, while those of the input and output vectors are r and p, respectively. Correspondingly, for the matrix multiplications in Eqs. (3.33) and (3.34) to be defined, the sizes of the coefficient matrices, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, should be $(n \times n)$, $(n \times r)$, $(p \times n)$, and $(p \times r)$, respectively. The coefficient matrices in Example 3.3 were all *constant*, i.e. they were not varying with time. Such a state-space representation in which all the coefficient matrices are constants is said to be *time-invariant.* In general, there are linear systems with coefficient matrices that are functions of time. Such state-space representations are said to be linear, but *time-varying*. Let us take another example of a *linear*, *time-invariant* state-space representation, which is a little more difficult to derive than the state-space representation of Example 3.3.

Example 3.4

Re-consider the electrical network presented in Example 2.8 whose governing differential equations are as follows:

$$R_1 i_1(t) + R_3[i_1(t) - i_2(t)] = e(t) \tag{3.35}$$

$$Li_2^{(2)}(t) + [(R_1R_3 + R_1R_2 + R_2R_3)/(R_1 + R_3)]i_2^{(1)}(t) + (1/C)i_2(t) = [R_3/(R_1 + R_3)]e^{(1)}(t) \tag{3.36}$$

If the input is the applied voltage, $e(t)$, and the output, $y_1(t)$, is the current in the resistor R_3 (given by $i_1(t) - i_2(t)$) when the switch S is closed (see Figure 2.19), we have to find a state-space representation of the system. Looking at Eq. (3.36), we find that the time derivative of the input appears on the right-hand side. For a linear, time-invariant state-space form of Eqs. (3.33) and (3.34), the state variables must be selected in such a way that the time derivative of the input, $e^{(1)}(t)$, vanishes from the state and output equations. One possible choice of state variables which accomplishes this is the following:

$$x_1(t) = i_2(t) \tag{3.37}$$

$$x_2(t) = i_2^{(1)}(t) - R_3 e(t)/[L(R_1 + R_3)] \tag{3.38}$$

Then the first state-equation is obtained by substituting Eq. (3.37) into Eq. (3.38), and expressed as

$$x_1^{(1)}(t) = x_2(t) + R_3 e(t)/[L(R_1 + R_3)] \tag{3.39}$$

Substitution of Eqs. (3.37) and (3.38) into Eq. (3.36) yields the second state-equation, given by

$$\begin{aligned} x_2^{(1)}(t) = &-[(R_1R_3 + R_1R_2 + R_2R_3)/[L(R_1 + R_3)] \\ &\times [x_2(t) + R_3 e(t)/L(R_1 + R_3)] - x_1(t)/(LC) \end{aligned} \tag{3.40}$$

The output equation is given by using Eq. (3.35) as follows:

$$\begin{aligned} y_1(t) &= i_1(t) - i_2(t) = [e(t) + R_3 x_1(t)]/(R_1 + R_3) - x_1(t) \\ &= -R_1 x_1(t)/(R_1 + R_3) + e(t)/(R_1 + R_3) \end{aligned} \tag{3.41}$$

In the matrix notation, Eqs. (3.39)–(3.41) are expressed as

$$\begin{aligned} \begin{bmatrix} x_1^{(1)}(t) \\ x_2^{(1)}(t) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -1/(LC) & -[(R_1R_3 + R_1R_2 + R_2R_3)/[L(R_1 + R_3)] \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \\ &+ \begin{bmatrix} R_3/[L(R_1 + R_3)] \\ -[R_3(R_1R_3 + R_1R_2 + R_2R_3)/[L(R_1 + R_3)]^2 \end{bmatrix} e(t) \end{aligned} \tag{3.42}$$

$$y_1(t) = \begin{bmatrix} -R_1/(R_1 + R_3) & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_1(t) \end{bmatrix} + 1/(R_1 + R_3) e(t) \tag{3.43}$$

Comparing Eqs. (3.42) and (3.43) with Eqs. (3.33) and (3.34), we can find the constant coefficient matrices, **A**, **B**, **C**, **D**, of the system, with the input vector, $\mathbf{u}(t) = e(t)$, and the output vector, $\mathbf{y}(t) = y_1(t)$.

If we compare Examples 3.3 and 3.4, it is *harder* to select the state variables in Example 3.4 due to the presence of the time derivative of the input in the governing differential equation. A general linear (or nonlinear) system may have several higher-order time derivatives of the input in its governing differential equations (such as Eq. (2.4)). To simplify the selection of state variables in such cases, it is often useful to first draw a *schematic diagram* of the governing differential equations. The schematic diagram is drawn using elements similar to those used in the *block diagram* of a system. These elements are the *summing-junction* (which adds two or more signals with appropriate

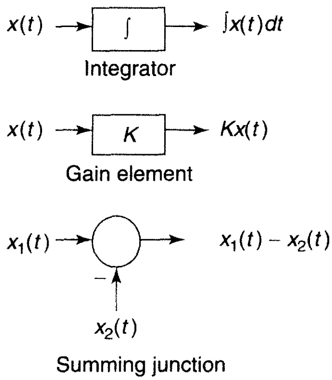

Figure 3.2 The state-space schematic diagram elements

signs), and the two transfer elements, namely the *gain element* (which multiplies a signal by a constant), and the *integrator* (which integrates a signal). The *arrows* are used to indicate the direction in which the signals are flowing into these elements. Figure 3.2 shows what the schematic diagram elements look like.

Let us use the schematic diagram approach to find another state-space representation for the system in Example 3.4.

Example 3.5

The system of Example 3.4 has two governing equations, Eqs. (3.35) and (3.36). While Eq. (3.35) is an *algebraic* equation (i.e. a *zero* order differential equation), Eq. (3.36) is a second order differential equation. Let us express Eq. (3.36) in terms of a *dummy variable* (so called because it is neither a state variable, an input, nor output) $z(t)$, such that

$$z^{(2)}(t) + [(R_1R_3 + R_1R_2 + R_2R_3)/[L(R_1 + R_3)]z^{(1)}(t) + 1/(LC)z(t) = e(t) \tag{3.44}$$

where

$$i_2(t) = R_3/[L(R_1 + R_3)]z^{(1)}(t) \tag{3.45}$$

We have split Eq. (3.36) into Eqs. (3.44) and (3.45) because we want to eliminate the time derivative of the input, $e^{(1)}(t)$, from the state-equations. You may verify that substituting Eq. (3.45) into Eq. (3.44) yields the original differential equation, Eq. (3.36). The schematic diagram of Eqs. (3.44) and (3.45) is drawn in Figure 3.3. Furthermore, Figure 3.3 uses Eq. (3.35) to represent the output, $y_1(t) = i_1(t) - i_2(t)$. Note the similarity between a block diagram, such as Figure 2.1, and a schematic diagram. Both have the inputs coming in from the left, and the outputs going out at the right. The difference between a block diagram and a schematic diagram is that, while the former usually represents the input-output relationship as a transfer function (or transfer matrix) in the Laplace domain, the latter represents

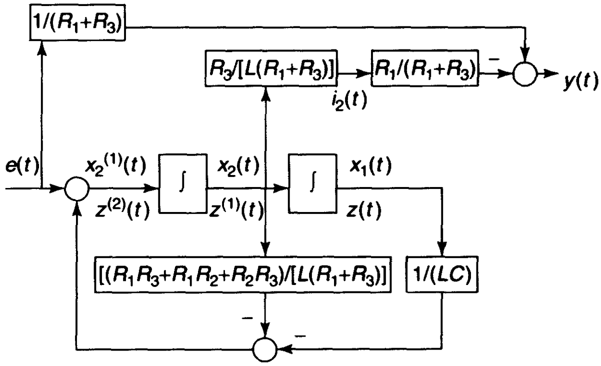

Figure 3.3 Schematic diagram for the electrical system of Example 3.4

the same relationship as a set of differential equations in time. Note that the number of integrators in a schematic diagram is equal to the order of the system.

A state-space representation of the system can be obtained from Figure 3.3 by choosing the *outputs of the integrators* as state variables, as shown in Figure 3.3. Then using the fact that the output from the *second integrator* from the left, $x_1(t)$, is the time integral of its input, $x_2(t)$, the first state-equation is given by

$$x_1^{(1)}(t) = x_2(t) \tag{3.46}$$

The second state-equation is obtained seeing what is happening at the *first summing junction* from the left. The *output* of that summing junction is the *input* to the *first integrator* from left, $x_2^{(1)}(t)$, and the two signals being added at the summing junction are $e(t)$ and $-x_1(t)/(LC) - [(R_1R_3 + R_1R_2 + R_2R_3)/[L(R_1 + R_3)]x_2(t)$. Therefore, the second state-equation is given by

$$x_2^{(1)}(t) = -x_1(t)/(LC) - [(R_1R_3 + R_1R_2 + R_2R_3)/[L(R_1 + R_3)]x_2(t) + e(t) \tag{3.47}$$

The output equation is obtained by expressing the output, $y_1(t) = i_1(t) - i_2(t)$, in terms of the state variables. Before relating the output to the state variables, we should express each state variable in terms of the physical quantities, $i_1(t)$, $i_2(t)$, and $e(t)$. We see from Figure 3.3 that $x_2(t) = z^{(1)}(t)$; thus, from Eq. (3.45), it follows that

$$x_2(t) = L(R_1 + R_3)i_2(t)/R_3 \tag{3.48}$$

Then, substitution of Eq. (3.48) into Eq. (3.47) yields

$$x_1(t) = LC[e(t) - i_2(t)(R_1R_3 + R_1R_2 + R_2R_3)/R_3 - L(R_1 + R_3)i_2^{(1)}(t)/R_3] \tag{3.49}$$

Using the algebraic relationship among $i_1(t)$, $i_2(t)$ and $e(t)$ by Eq. (3.35), we can write the output equation as follows:

$$y_1(t) = -R_1R_3/[L(R_1+R_3)^2]x_2(t) + e(t)/(R_1+R_3) \quad (3.50)$$

In matrix form, the state-space representation is given by

$$\begin{bmatrix} x_1^{(1)}(t) \\ x_2^{(1)}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1/(LC & -(R_1R_3+R_1R_2+R_2R_3)/[L(R_1+R_3)] \end{bmatrix}$$

$$\times \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e(t) \quad (3.51)$$

$$y_1(t) = [0 - R_1R_3/\{L(R_1+R_3)^2\}] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + 1/(R_1+R_3)e(t) \quad (3.52)$$

Note the difference in the state-space representations of the same system given by Eqs. (3.42), (3.43) and by Eqs. (3.51), (3.52). Also, note that *another* state-space representation could have been obtained by numbering the state variables in Figure 3.3 starting from the *left* rather than from the *right*, which we did in Example 3.5.

Example 3.6

Let us find a state-space representation using the schematic diagram for the system with input, $u(t)$, and output, $y(t)$, described by the following differential equation:

$$\begin{aligned} &y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \cdots + a_1y^{(1)}(t) + a_0y(t) \\ &\quad = b_nu^{(n)}(t) + b_{n-1}u^{(n-1)}(t) + \cdots + b_1u^{(1)}(t) + b_0u(t) \end{aligned} \quad (3.53)$$

Since the right-hand side of Eq. (3.53) contains time derivatives of the input, we should introduce a dummy variable, $z(t)$, in a manner similar to Example 3.5, such that

$$z^{(n)}(t) + a_{n-1}z^{(n-1)}(t) + \cdots + a_1z^{(1)}(t) + a_0z(t) = u(t) \quad (3.54)$$

and

$$b_nz^{(n)}(t) + b_{n-1}z^{(n-1)}(t) + \cdots + b_1z^{(1)}(t) + b_0z(t) = y(t) \quad (3.55)$$

Figure 3.4 shows the schematic diagram of Eqs. (3.54) and (3.55). Note that Figure 3.4 has n integrators arranged in a series.

As in Example 3.5, let us choose the state variables to be the integrator outputs, and number them beginning from the right of Figure 3.4. Then the state-equations are as follows:

$$x_1^{(1)}(t) = x_2(t) \quad (3.56a)$$

$$x_2^{(1)}(t) = x_3(t) \quad (3.56b)$$

$$\vdots$$

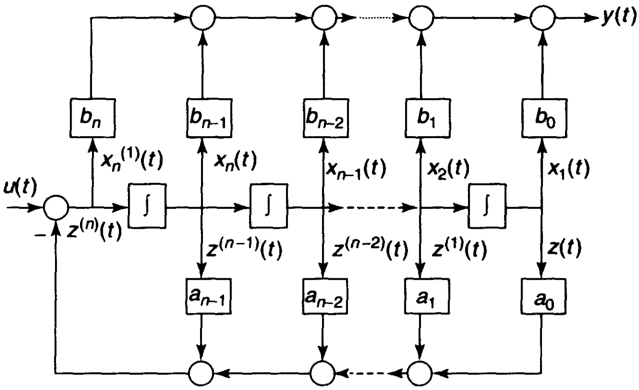

Figure 3.4 Schematic diagram for the controller companion form of the system in Example 3.6

$$x_{n-1}^{(1)}(t) = x_n(t) \tag{3.56c}$$

$$x_n^{(1)}(t) = -[a_{n-1}x_n(t) + a_{n-2}x_{n-1}(t) + \cdots + a_0x_1(t) - u(t)] \tag{3.56d}$$

The output equation is obtained by substituting the definitions of the state variables, namely, $x_1(t) = z(t)$, $x_2(t) = z^{(1)}(t), \ldots, x_n(t) = z^{(n-1)}(t)$ into Eq. (3.55), thereby yielding

$$y(t) = b_0x_1(t) + b_1x_2(t) + \cdots + b_{n-1}x_n(t) + b_nx_n^{(1)}(t) \tag{3.57}$$

and substituting Eq. (3.56d) into Eq. (3.57), the output equation is expressed as follows:

$$y(t) = (b_0 - a_0b_n)x_1(t) + (b_1 - a_1b_n)x_2(t) + \cdots + (b_{n-1} - a_{n-1}b_n)x_n(t) + b_nu(t) \tag{3.58}$$

The matrix form of the state-equations is the following:

$$\begin{bmatrix} x_1^{(1)}(t) \\ x_2^{(1)}(t) \\ \vdots \\ x_{n-1}^{(1)}(t) \\ x_n^{(1)}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(t) \tag{3.59}$$

and the output equation in matrix form is as follows:

$$y(t) = [(b_0 - a_0b_n)\ (b_1 - a_1b_n) \ldots (b_{n-1} - a_{n-1}b_n)] \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + b_nu(t) \tag{3.60}$$

Comparing Eqs. (3.59) and (3.60) with the matrix equations Eqs. (3.33) and (3.34), respectively, we can easily find the coefficient matrices, **A**, **B**, **C**, **D**, of the state-space representation. Note that the matrix **A** of Eq. (3.59) has a particular structure: all elements except the *last row* and the *superdiagonal* (i.e. the diagonal *above* the main diagonal) are *zeros*. The superdiagonal elements are all *ones*, while the last row consists of the coefficients with a negative sign, $-a_0, -a_1, \ldots, -a_{n-1}$. Taking the Laplace transform of Eq. (3.53), you can verify that the coefficients $a_0, a_1, \ldots, a_{n-1}$ are the *coefficients of the characteristic polynomial* of the system (i.e. the denominator polynomial of the transfer function, $Y(s)/U(s)$) given by $s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$. The matrix **B** of Eq. (3.59) has all elements zeros, except the last row (which equals 1). Such a state-space representation has a name: *the controller companion form*. It is thus called because it has a special place in the design of controllers, which we will see in Chapter 5.

Another *companion form*, called the *observer companion form*, is obtained as follows for the system obeying Eq. (3.53). In Eq. (3.53) the terms involving derivatives of $y(t)$ and $u(t)$ of the same order are collected, and the equation is written as follows:

$$[y^{(n)}(t) - b_n u^{(n)}(t)] + [a_{n-1}y^{(n-1)}(t) - b_{n-1}u^{(n-1)}(t)] \\ + \cdots + [a_1 y^{(1)}(t) - b_1 u^{(1)}(t)] + [a_0 y(t) - b_0 u(t)] = 0 \qquad (3.61)$$

On taking the Laplace transform of Eq. (3.61) subject to zero initial conditions, we get the following:

$$s^n[Y(s) - b_n U(s)] + s^{n-1}[a_{n-1}Y(s) - b_{n-1}U(s)] \\ + \cdots + s[a_1 Y(s) - b_1 U(s)] + [a_0 Y(s) - b_0 U(s)] = 0 \qquad (3.62)$$

Dividing Eq. (3.61) by s^n leads to

$$Y(s) = b_n U(s) + [b_{n-1}U(s) - a_{n-1}Y(s)]/s + \cdots + [b_1 U(s) - a_1 Y(s)]/s^{n-1} \\ + [b_0 U(s) - a_0 Y(s)]/s^n \qquad (3.63)$$

We can draw a schematic diagram for Eq. (3.63), using the fact that the multiplication factor $1/s$ in the Laplace domain represents an integration in time. Therefore, according to Eq. (3.63), $[b_{n-1}U(s) - a_{n-1}Y(s)]$ must pass through *one integrator* before contributing to the output, $Y(s)$. Similarly, $[b_1 U(s) - a_1 Y(s)]$ must pass through $(n-1)$ integrators, and $[b_0 U(s) - a_0 Y(s)]$ through n integrators in the schematic diagram. Figure 3.5 shows the schematic diagram of Eq. (3.63).

On comparing Figures 3.4 and 3.5, we see that both the figures have a series of n integrators, but the feedback paths from the output, $y(t)$, to the integrators are in *opposite* directions in the two figures. If we select the outputs of the integrators as state variables beginning from the *left* of Figure 3.5, we get the following state-equations:

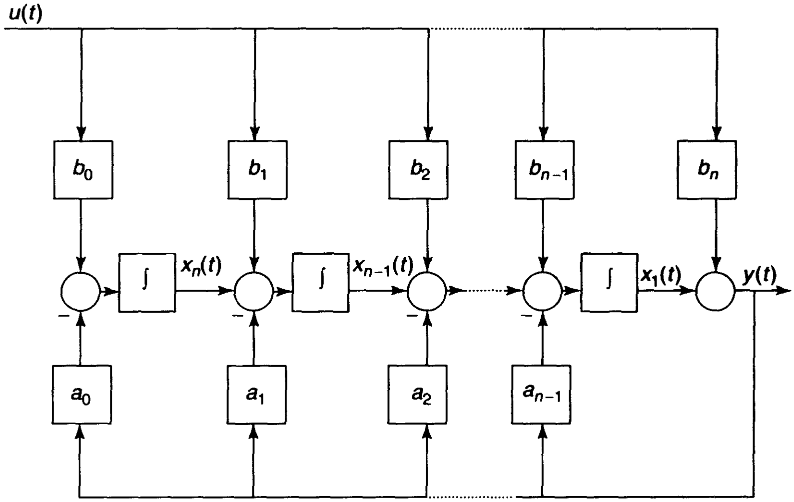

Figure 3.5 Schematic diagram for the observer companion form of the system in Example 3.6

$$
\begin{aligned}
x_1^{(1)}(t) &= -a_0 x_n(t) + (b_0 - a_0 b_n)u(t) \\
x_2^{(1)}(t) &= x_1(t) - a_1 x_n(t) + (b_1 - a_1 b_n)u(t) \\
&\vdots \\
x_{n-1}^{(1)}(t) &= x_{n-2}(t) - a_{n-2} x_n(t) + (b_{n-2} - a_{n-2} b_n)u(t) \\
x_n^{(1)}(t) &= -a_{n-1} x_n(t) + (b_{n-1} - a_{n-1} b_n)
\end{aligned}
\tag{3.64}
$$

and the output equation is

$$y(t) = x_n(t) + b_n u(t) \tag{3.65}$$

Therefore, the state coefficient matrices, **A**, **B**, **C**, and **D**, of the observer companion form are written as follows:

$$
\mathbf{A} = \begin{bmatrix}
0 & 0 & \ldots & 0 & -a_0 \\
1 & 0 & \ldots & 0 & -a_1 \\
0 & 1 & \ldots & 0 & -a_2 \\
\cdot & \cdot & \ldots & \cdot & \cdot \\
\cdot & \cdot & \ldots & \cdot & \cdot \\
\cdot & \cdot & \ldots & \cdot & \cdot \\
0 & 0 & \ldots & 1 & -a_{n-2} \\
0 & 0 & \ldots & 0 & -a_{n-1}
\end{bmatrix}; \quad
B = \begin{bmatrix}
(b_0 - a_0 b_n) \\
(b_1 - a_1 b_n) \\
(b_2 - a_2 b_n) \\
\cdot \\
\cdot \\
\cdot \\
(b_{n-2} - a_{n-2} b_n) \\
(b_{n-1} - a_{n-1} b_n)
\end{bmatrix}
$$

$$\mathbf{C} = [0 \quad 0 \quad \ldots \quad 0 \quad 1]; \quad \mathbf{D} = b_n \tag{3.66}$$

Note that the **A** matrix of the observer companion form is the transpose of the **A** matrix of the controller companion form. Also, the **B** matrix of the observer companion form is the transpose of the **C** matrix of the controller companion form, and vice versa. The **D** matrices of the both the companion forms are the same. Now we know why these state-space representations are called *companion forms*: they can be obtained from one another merely by taking the transpose of the coefficient matrices. The procedure used in this example for obtaining the companion forms of a single-input, single-output system can be extended to multi-variable systems.

Thus far, we have only considered examples having single inputs. Let us take up an example with multi-inputs.

Example 3.7

Consider the electrical network for an amplifier-motor shown in Figure 3.6. It is desired to change the angle, $\theta(t)$, and angular velocity, $\theta^{(1)}(t)$, of a load attached to the motor by changing the input voltage, $e(t)$, in the presence of a torque, $T_L(t)$, applied by the load on the motor. The governing differential equations for the amplifier-motor are the following:

$$Li^{(1)}(t) + (R + R_0)i(t) + a\theta^{(1)}(t) = K_A e(t) \tag{3.67}$$

$$J\theta^{(2)}(t) + b\theta^{(1)}(t) - ai(t) = -T_L(t) \tag{3.68}$$

where J, R, L, and b are the moment of inertia, resistance, self-inductance, and viscous damping-coefficient of the motor, respectively, and a is a machine constant. R_0 and K_A are the resistance and voltage amplification ratio of the amplifier.

Since the loading torque, $T_L(t)$, acts as a disturbance to the system, we can consider it as an additional input variable. The input vector is thus given by $\mathbf{u}(t) = [e(t); T_L(t)]^T$. The output vector is given by $\mathbf{y}(t) = [\theta(t); \theta^{(1)}(t)]^T$. We see from Eqs. (3.67) and (3.68) that the system is of third order. Hence, we need three state variables for the state-space representation of the system. Going by the desirable convention of choosing state variables to be physical quantities, let us select the state variables as $x_1(t) = \theta(t)$, $x_2(t) = \theta^{(1)}(t)$, and $x_3(t) = i(t)$. Then the state-equations can be written as follows:

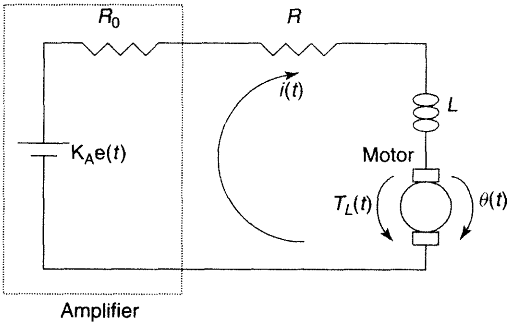

Figure 3.6 Amplifier-motor circuit of Example 3.7