

Since a system's characteristics do not change when we express the same system by different state-space representations, the linear state transformations are also called *similarity transformations*. Let us now see how we can obtain information about a system's characteristics – locations of poles, performance, stability, etc. – from its state-space representation.

3.3 System Characteristics from State-Space Representation

In Chapter 2, we defined the characteristics of a system by its characteristic equation, whose roots are the poles of the system. We also saw how the locations of the poles indicate a system's performance – such as natural frequency, damping factor, system type – as well as whether the system is stable. Let us see how a system's characteristic equation can be derived from its state-space representation.

The characteristic equation was defined in Chapter 2 to be the denominator polynomial of the system's transfer function (or transfer matrix) equated to zero. Hence, we should first obtain an expression for the transfer matrix in terms of the state-space coefficient matrices, **A**, **B**, **C**, **D**. Recall that the transfer matrix is obtained by taking the Laplace transform of the governing differential equations, for zero initial conditions. Taking the Laplace transform of both sides of the matrix state-equation, Eq. (3.33), assuming zero initial conditions (i.e. $\mathbf{x}(0) = \mathbf{0}$) yields the following result:

$$s\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s) \quad (3.99)$$

where $\mathbf{X}(s) = \mathcal{L}[\mathbf{x}(t)]$, and $\mathbf{U}(s) = \mathcal{L}[\mathbf{u}(t)]$. Rearranging Eq. (3.99), we can write

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{B}\mathbf{U}(s) \quad (3.100)$$

or

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) \quad (3.101)$$

Similarly, taking the Laplace transform of the output equation, Eq. (3.34), with $\mathbf{Y}(s) = \mathcal{L}[\mathbf{y}(t)]$, yields

$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s) \quad (3.102)$$

Substituting Eq. (3.101) into Eq. (3.102) we get

$$\mathbf{Y}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) + \mathbf{D}\mathbf{U}(s) = [\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]\mathbf{U}(s) \quad (3.103)$$

From Eq. (3.103), it is clear that the transfer matrix, $\mathbf{G}(s)$, defined by $\mathbf{Y}(s) = \mathbf{G}(s)\mathbf{U}(s)$, is the following:

$$\mathbf{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} \quad (3.104)$$

Equation (3.104) tells us that the transfer matrix is a sum of the *rational matrix* (i.e. a matrix whose elements are ratios of polynomials in s), $\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$, and the

matrix $\mathbf{D}$. Thus, $\mathbf{D}$ represents a *direct connection* between the input, $\mathbf{U}(s)$, and the output, $\mathbf{Y}(s)$, and is called the *direct transmission matrix*. Systems having $\mathbf{D} = \mathbf{0}$ are called *strictly proper*, because the numerator polynomials of the elements of $\mathbf{G}(s)$ are smaller in degree than the corresponding denominator polynomials (see the discussion following Eq. (2.1) for the definition of strictly proper single variable systems). In Example 2.28, we had obtained the characteristic equation of a multivariable system from the denominator polynomial of $\mathbf{G}(s)$. Hence, the characteristic polynomial of the system must be related to the denominator polynomial resulting from the matrix, $\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$.

Example 3.11

For a linear system described by the following state coefficient matrices, let us determine the transfer function and the characteristic equation:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad \mathbf{C} = [1 \quad 1]; \quad \mathbf{D} = 0 \quad (3.105)$$

The inverse $(s\mathbf{I} - \mathbf{A})^{-1}$ is calculated as follows:

$$(s\mathbf{I} - \mathbf{A})^{-1} = \text{adj}(s\mathbf{I} - \mathbf{A})/|(s\mathbf{I} - \mathbf{A})| \quad (3.106)$$

where the *determinant* $|(s\mathbf{I} - \mathbf{A})|$ is given by

$$|(s\mathbf{I} - \mathbf{A})| = \begin{vmatrix} (s-1) & -2 \\ 2 & (s-1) \end{vmatrix} = (s-1)^2 + 4 = s^2 - 2s + 5 \quad (3.107)$$

and the *adjoint*, $\text{adj}(s\mathbf{I} - \mathbf{A})$, is given by

$$\text{adj}(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} (s-1) & -2 \\ 2 & (s-1) \end{bmatrix}^T = \begin{bmatrix} (s-1) & 2 \\ -2 & (s-1) \end{bmatrix} \quad (3.108)$$

(See Appendix B for the definitions of the inverse, adjoint, and determinant.) Substituting Eqs. (3.107) and (3.108) into Eq. (3.106) we get

$$(s\mathbf{I} - \mathbf{A})^{-1} = [1/(s^2 - 2s + 5)] \begin{bmatrix} (s-1) & 2 \\ -2 & (s-1) \end{bmatrix} \quad (3.109)$$

Then the transfer matrix is calculated as follows:

$$\begin{aligned} \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1} &= [1/(s^2 - 2s + 5)][1 \quad 1] \begin{bmatrix} (s-1) & 2 \\ -2 & (s-1) \end{bmatrix} \\ &= [1/(s^2 - 2s + 5)][(s-3) \quad (s+1)] \end{aligned} \quad (3.110)$$

$$\begin{aligned} \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} &= [1/(s^2 - 2s + 5)][(s-3); \quad (s+1)] \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= (s-3)/(s^2 - 2s + 5) \end{aligned} \quad (3.111)$$

The conversion of a system's state-space representation into its transfer matrix is easily carried out with the MATLAB Control System Toolbox's (CST) LTI object function *tf* as follows:

```
>>sys = tf(sys) <enter>
```

Example 3.12

Let us convert the state-space representation of the aircraft longitudinal dynamics (Example 3.10) given by Eqs. (3.96) and (3.97) into the transfer matrix of the system, as follows:

```
>>sys1=tf(sys1) <enter>
```

```
Transfer function from input 1 to output...
          -10s^2-29.25s-1.442
```

```
#1: -----
      s^4+6.245s^3+19.9s^2+1.003s+0.7033
```

```
          -10s^3-29.25s^2-1.442s-5.293e-016
#2: -----
      s^4+6.245s^3+19.9s^2+1.003s+0.7033
```

```
Transfer function from input 2 to output...
          0.0002s+0.0022
```

```
#1: -----
      s^4+6.245s^3+19.9s^2+1.003s+0.7033
```

```
          0.0002s^2+0.0022s+3.671e-019
#2: -----
      s^4+6.245s^3+19.9s^2+1.003s+0.7033
```

Note that the single-input, single-output system of Example 3.11, the transfer function has a denominator polynomial $s^2 - 2s + 5$, which is also the *characteristic polynomial* of the system (see Chapter 2). The denominator polynomial is equal to $|(s\mathbf{I} - \mathbf{A})|$ (Eq. (3.107)). Thus, the poles of the transfer function are the roots of the characteristic equation, $|(s\mathbf{I} - \mathbf{A})| = 0$. This is also true for the multivariable system of Example 3.12, where all the elements of the transfer matrix have the same denominator polynomial. Using linear algebra, the characteristic equation of a general, linear time-invariant system is obtained from the following *eigenvalue problem* for the system:

$$\mathbf{A}\mathbf{v}_k = \lambda_k \mathbf{v}_k \quad (3.112)$$

where λ_k is the k th *eigenvalue* of the matrix $\mathbf{A}$, and $\mathbf{v}_k$ is the *eigenvector* associated with the eigenvalue, λ_k (see Appendix B). Equation (3.112) can be written as follows:

$$(\lambda\mathbf{I} - \mathbf{A})\mathbf{v} = \mathbf{0} \quad (3.113)$$

For the nontrivial solution of Eq. (3.113) (i.e. $\mathbf{v} \neq \mathbf{0}$), the following must be true:

$$|(\lambda \mathbf{I} - \mathbf{A})| = 0 \quad (3.114)$$

Equation (3.114) is another way of writing the characteristic equation, whose the roots are the eigenvalues, λ . Hence, the *poles of the transfer matrix* are the *same as the eigenvalues of the matrix, A*. Since **A** contains information about the characteristic equation of a system, it influences all the properties such as stability, performance and robustness of the system. For this reason, **A** is called the system's *state-dynamics matrix*.

Example 3.13

For the state-space representation of the electrical network derived in Examples 3.4 and 3.5, let us substitute the numerical values from Example 2.8 ($R_1 = R_3 = 10$ ohms, $R_2 = 25$ ohms, $L = 1$ henry, $C = 10^{-6}$ farad) and calculate the transfer functions and eigenvalues for both the state-space representations.

The state-space representation of Example 3.4 yields the following state coefficient matrices:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -10^6 & -30 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 0.5 \\ -15 \end{bmatrix}; \quad \mathbf{C} = [-0.5 \quad 0]; \quad \mathbf{D} = 0.05 \quad (3.115)$$

while the state-space representation of Example 3.5 has the following coefficient matrices:

$$\mathbf{A}' = \begin{bmatrix} 0 & 1 \\ -10^6 & -30 \end{bmatrix}; \quad \mathbf{B}' = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad \mathbf{C}' = [0 \quad -0.25]; \quad \mathbf{D}' = 0.05 \quad (3.116)$$

Either using Eq. (3.104) by hand, or using the CST LTI object function, *tf*, we can calculate the respective transfer functions as follows:

```
>>A=[0 1;-1e6 -30]; B=[0.5; -15]; C=[-0.5 0]; D=0.05;sys1=ss(A,B,C,D);
sys1=tf(sys1) <enter>
```

```
Transfer function:
0.05s^2+1.25s+5e004
-----
s^2+30s+1e006
```

```
>> A=[0 1;-1e6 -30]; B=[0; 1]; C=[0 -0.25]; D=0.05;sys2=ss(A,B,C,D);
sys2=tf(sys1) <enter>
```

```
Transfer function:
0.05s^2+1.25s+5e004
-----
s^2+30s+1e006
```

Note that the two transfer functions are identical, as expected, because the two state-space representations are for the *same system*. The characteristic equation is

obtained by equating the denominator polynomial to zero, i.e. $s^2 + 30s + 10^6 = 0$. Solving the characteristic equation, we get the poles of the system as follows:

```
>>roots([1 30 1e6]) <enter>
```

```
ans =  
-1.5000e+001 +9.9989e+002i  
-1.5000e+001 -9.9989e+002i
```

which agree with the result of Example 2.8. These poles should be the same as the eigenvalues of the matrix, $\mathbf{A}(=\mathbf{A}')$, obtained using the intrinsic MATLAB function *eig* as follows:

```
>>eig([0 1;-1e6 -30]) <enter>
```

```
ans =  
-1.5000e+001 +9.9989e+002i  
-1.5000e+001 -9.9989e+002i
```

Example 3.13 shows that the system's characteristics are unchanged by using different state-space representations.

Example 3.14

Consider the following two-input, two-output *turbo-generator* system [2]:

$$\mathbf{A} = \begin{bmatrix} -18.4456 & 4.2263 & -2.2830 & 0.2260 & 0.4220 & -0.0951 \\ -4.0977 & -6.0706 & 5.6825 & -0.6966 & -1.2246 & 0.2873 \\ 1.4449 & 1.4336 & -2.6477 & 0.6092 & 0.8979 & -0.2300 \\ -0.0093 & 0.2302 & -0.5002 & -0.1764 & -6.3152 & 0.1350 \\ -0.0464 & -0.3489 & 0.7238 & 6.3117 & -0.6886 & 0.3645 \\ -0.0602 & -0.2361 & 0.2300 & 0.0915 & -0.3214 & -0.2087 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} -0.2748 & 3.1463 \\ -0.0501 & -9.3737 \\ -0.1550 & 7.4296 \\ 0.0716 & -4.9176 \\ -0.0814 & -10.2648 \\ 0.0244 & 13.7943 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 0.5971 & -0.7697 & 4.8850 & 4.8608 & -9.8177 & -8.8610 \\ 3.1013 & 9.3422 & -5.6000 & -0.7490 & 2.9974 & 10.5719 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (3.117)$$

The eigenvalues and the associated natural frequencies and damping factors of the system are found by using the MATLAB command *damp(A)* as follows:

```
>>damp(A) <enter>
```

Eigenvalue	Damping	Freq. (rad/sec)
-2.3455e-001	1.0000e+000	2.3455e-001
-3.4925e-001+6.3444e+000i	5.4966e-002	6.3540e+000
-3.4925e-001-6.3444e+000i	5.4966e-002	6.3540e+000
-1.0444e+000	1.0000e+000	1.0444e+000
-1.0387e+001	1.0000e+000	1.0387e+001
-1.5873e+001	1.0000e+000	1.5873e+001

Note that there are four real eigenvalues, and a pair of complex conjugate eigenvalues. All the eigenvalues (i.e. poles) have negative real parts, implying an *asymptotically stable system* from the stability criteria of Chapter 2. Also from Chapter 2, the damping factors associated with all real eigenvalues with negative real parts are 1.0, since such eigenvalues represent *exponentially decaying* responses. Only complex conjugate eigenvalues have damping factors less than 1.0. These eigenvalues represent an oscillatory response. If it were possible to *decouple* the state-equations by the use of a state transformation, such that the each of the transformed state-equations is in terms of *only one* state variable, then each eigenvalue would represent a particular *mode* in which the system can respond. Hence, there are *six modes* in this sixth order system, consisting of four real (or *first order*) modes, and a *second order* mode defined by a pair of complex conjugate eigenvalues. Note that the second order mode has a relatively small damping factor (0.055). The transfer matrix, $\mathbf{G}(s)$, defined by $\mathbf{Y}(s) = \mathbf{G}(s)\mathbf{U}(s)$, of this two-input, two-output system, is written as follows:

$$\mathbf{G}(s) = [\mathbf{G}_1(s) \quad \mathbf{G}_2(s)] \quad (3.118)$$

where $\mathbf{Y}(s) = \mathbf{G}_1(s)U_1(s) + \mathbf{G}_2(s)U_2(s)$, with $U_1(s)$ and $U_2(s)$ being the two inputs. $\mathbf{G}_1(s)$ and $\mathbf{G}_2(s)$ are usually obtained using the CST LTI objects *ss* and *tf* as follows:

```
>>sys=ss(A,B,C,D); syst=tf(syst) <enter>
```

```
Transfer function from input 1 to output...
```

```
0.04829s^5+1.876s^4+1.949s^3-1228s^2-5762s-2385
```

```
#1:-----
s^6+28.24s^5+258.3s^4+1468s^3+8214s^2+8801s+1631
```

```
-0.4919s^5+9.483s^4-49.05s^3+551.6s^2-939.6s+907.6
```

```
#2:-----
s^6+28.24s^5+258.3s^4+1468s^3+8214s^2+8801s+1631
```

```
Transfer function from input 2 to output...
```

```
0.02915s^5+1.289s^4-0.3041s^3-2.388e004s^2-8.29e005s
-9.544e005
```