4

Solving the State-Equations

4.1 Solution of the Linear Time Invariant State Equations

We learnt in Chapter 3 how to represent the governing differential equation of a system by a set of first order differential equations, called state-equations, whose number is equal to the order of the system. Before we can begin designing a control system based on the state-space approach, we must be able to solve the state-equations. To see how the state-equations are solved, let us consider the following single first order differential equation:

$$x^{(1)}(t) = ax(t) + bu(t) \tag{4.1}$$

where $x(t)$ is the state variable, $u(t)$ is the input, and a and b are the constant coefficients. Equation (4.1) represents a first order system. Let us try to solve this equation for $t > t_0$ with the *initial condition*, $x(t_0) = x_0$. (Note that since the differential equation, Eq. (4.1), is of first order we need only *one* initial condition to obtain its solution). The solution to Eq. (4.1) is obtained by multiplying both sides of the equation by $\exp\{-a(t-t_0)\}$ and re-arranging the resulting equation as follows:

$$\exp\{-a(t-t_0)\}x^{(1)}(t) - \exp\{-a(t-t_0)\}ax(t) = \exp\{-a(t-t_0)\}bu(t) \tag{4.2}$$

We recognize the term on the left-hand side of Eq. (4.2) as $d/dt[\exp\{-a(t-t_0)\}x(t)]$. Therefore, Eq. (4.2) can be written as

$$d/dt[\exp\{-a(t-t_0)\}x(t)] = \exp\{-a(t-t_0)\}bu(t) \tag{4.3}$$

Integrating both sides of Eq. (4.3) from t_0 to t, we get

$$\exp\{-a(t-t_0)\}x(t) - x(t_0) = \int_{t_0}^{t} \exp\{-a(\tau-t_0)\}bu(\tau)d\tau \tag{4.4}$$

Applying the initial condition, $x(t_0) = x_0$, and multiplying both sides of Eq. (4.4) by $\exp\{a(t-t_0)\}$, we get the following expression for the state variable, $x(t)$:

$$x(t) = \exp\{a(t-t_0)\}x_0 + \int_{t_0}^{t} e^{a(t-\tau)}bu(\tau)d\tau; \quad (t \geq t_0) \tag{4.5}$$

Note that Eq. (4.5) has two terms on the right-hand side. The first term, $\exp\{a(t - t_0)\}x_0$, depends upon the initial condition, x_0, and is called the *initial response* of the system. This will be the only term present in the response, $x(t)$, if the applied input, $u(t)$, is zero. The integral term on the right-hand side of Eq. (4.5) is independent of the initial condition, but depends upon the input. Note the similarity between this integral term and the *convolution integral* given by Eq. (2.120), which was derived as the response of a linear system to an arbitrary input by linearly superposing the individual impulse responses. The lower limit of the integral in Eq. (4.5) is t_0 (instead of $-\infty$ in Eq. (2.120)), because the input, $u(t)$, *starts acting* at time t_0 onwards, and is assumed to be zero at all times $t < t_0$. (Of course, one could have an *ever-present* input, which starts acting on the system at $t = -\infty$; in that case, $t_0 = -\infty$). If the coefficient, a, in Eq. (4.1) is *negative*, then the system given by Eq. (4.1) is *stable* (why?), and the response given by Eq. (4.5) will reach a *steady-state* in the limit $t \to \infty$. Since the initial response of the stable system decays to zero in the limit $t \to \infty$, the integral term is the only term remaining in the response of the system in the steady-state limit. Hence, the integral term in Eq. (4.5) is called the *steady-state response* of the system. All the system responses to singularity functions with zero initial condition, such as the *step response* and the *impulse response*, are obtained form the steady-state response. Comparing Eqs. (2.120) and (4.5), we can say that for this first order system the *impulse response*, $g(t - t_0)$, is given by

$$g(t - t_0) = \exp\{a(t - t_0)\}b \tag{4.6}$$

You may verify Eq. (4.6) by deriving the impulse response of the first order system of Eq. (4.1) using the Laplace transform method of Chapter 2 for $u(t) = \delta(t - t_0)$ and $x(t_0) = 0$. The step response, $s(t)$, of the system can be obtained as the time integral of the impulse response (see Eqs. (2.104) and (2.105)), given by

$$s(t) = \int_{t_0}^{t} e^{a(t-\tau)} b d\tau = [\exp\{a(t - t_0)\} - 1]/a \tag{4.7}$$

Note that Eq. (4.7) can also be obtained directly from Eq. (4.5) by putting $u(t) = u_s(t - t_0)$ and $x(t_0) = 0$.

To find the response of a general system of order n, we should have a solution for each of the n state-equations in a form similar to Eq. (4.5). However, since the state-equations are usually *coupled*, their solutions cannot be obtained *individually*, but *simultaneously* as a *vector solution*, $\mathbf{x}(t)$, to the following matrix state-equation:

$$\mathbf{x}^{(1)}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{4.8}$$

Before considering the general matrix state-equation, Eq. (4.8), let us take the special case of a system having *distinct eigenvalues*. We know from Chapter 3 that for such systems, the state-equations can be *decoupled* through an appropriate state transformation. Solving decoupled state-equations is a simple task, consisting of individual application of Eq. (4.5) to each decoupled state-equation. This is illustrated in the following example.

Example 4.1

Consider a system with the following state-space coefficient matrices:

$$\mathbf{A} = \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{4.9}$$

Let us solve the state-equations for $t > 0$ with the following initial condition:

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{4.10}$$

The individual scalar state-equations can be expressed from Eq. (4.8) as follows:

$$x_1^{(1)}(t) = -3x_1(t) + u(t) \tag{4.11}$$

$$x_2^{(1)}(t) = -2x_2(t) - u(t) \tag{4.12}$$

where $x_1(t)$ and $x_2(t)$ are the state variables, and $u(t)$ is the input defined for $t \geq 0$. Since both Eqs. (4.11) and (4.12) are decoupled, they are solved independently of one another, and their solutions are given by Eq. (4.5) as follows:

$$x_1(t) = \mathrm{e}^{-3t} + \int_0^t \mathrm{e}^{-3(t-\tau)} u(\tau) d\tau; \quad (t \geq 0) \tag{4.13}$$

$$x_2(t) = -\int_0^t \mathrm{e}^{-2(t-\tau)} u(\tau) d\tau; \quad (t \geq 0) \tag{4.14}$$

Example 4.1 illustrates the ease with which the decoupled state-equations are solved. However, only systems with distinct eigenvalues can be decoupled. For systems having repeated eigenvalues, we must be able to solve the coupled state-equations given by Eq. (4.8).

To solve the general state-equations, Eq. (4.8), let us first consider the case when the input vector, $\mathbf{u}(t)$, is *always zero*. Then Eq. (4.8) becomes a *homogeneous* matrix state-equation given by

$$\mathbf{x}^{(1)}(t) = \mathbf{A}\mathbf{x}(t) \tag{4.15}$$

We are seeking the vector solution, $\mathbf{x}(t)$, to Eq. (4.15) subject to the initial condition, $\mathbf{x}(t_0) = \mathbf{x}_0$. The solution to the *scalar counterpart* of Eq. (4.15) (i.e. $x^{(1)}(t) = ax(t)$) is just the initial response given by $x(t) = \exp\{a(t - t_0)\}x_0$, which we obtain from Eq. (4.5) by setting $u(t) = 0$. Taking a hint from the scalar solution, let us write the vector solution to Eq. (4.15) as

$$\mathbf{x}(t) = \exp\{\mathbf{A}(t - t_0)\}\mathbf{x}(t_0) \tag{4.16}$$

In Eq. (4.16) we have introduced a strange beast, $\exp\{\mathbf{A}(t - t_0)\}$, which we will call *the matrix exponential of* $\mathbf{A}(t - t_0)$. This beast is somewhat like the Loch Ness monster, whose existence has been conjectured, but not proven. Hence, it is a figment of our imagination. Everybody has seen and used the scalar exponential, $\exp\{a(t - t_0)\}$, but talking about a *matrix* raised to the power of a *scalar*, e, appears to be stretching our credibility beyond its limits! Anyhow, since Eq. (4.16) tells us that the matrix exponential can help us in solving the general state-equations, let us see how this animal can be defined.

We know that the Taylor series expansion of the scalar exponential, $\exp\{a(t - t_0)\}$, is given by

$$\begin{aligned}\exp\{a(t - t_0)\} &= 1 + a(t - t_0) + a^2(t - t_0)^2/2! \\ &\quad + a^3(t - t_0)^3/3! + \cdots + a^k(t - t_0)^k/k! + \cdots \end{aligned} \tag{4.17}$$

Since the matrix exponential behaves exactly like the scalar exponential in expressing the solution to a first order differential equation, we conjecture that it must also have the same expression for its Taylor series as Eq. (4.17) with the scalar, a, replaced by the matrix, $\mathbf{A}$. Therefore, we *define* the matrix exponential, $\exp\{\mathbf{A}(t - t_0)\}$, as a matrix that has the following Taylor series expansion:

$$\begin{aligned}\exp\{\mathbf{A}(t - t_0)\} &= \mathbf{I} + \mathbf{A}(t - t_0) + \mathbf{A}^2(t - t_0)^2/2! \\ &\quad + \mathbf{A}^3(t - t_0)^3/3! + \cdots + \mathbf{A}^k(t - t_0)^k/k! + \cdots \end{aligned} \tag{4.18}$$

Equation (4.18) tells us that the matrix exponential is of the same size as the matrix $\mathbf{A}$. Our definition of $\exp\{\mathbf{A}(t - t_0)\}$ must satisfy the homogeneous matrix state-equation, Eq. (4.15), whose solution is given by Eq. (4.16). To see whether it does so, let us differentiate Eq. (4.16) with time, t, to yield

$$\mathbf{x}^{(1)}(t) = d/dt[\exp\{\mathbf{A}(t - t_0)\}\mathbf{x}(t_0)] = d/dt[\exp\{\mathbf{A}(t - t_0)\}]\mathbf{x}(t_0) \tag{4.19}$$

The term $d/dt[\exp\{\mathbf{A}(t - t_0)\}]$ is obtained by differentiating Eq. (4.17) with respect to time, t, as follows:

$$\begin{aligned}&d/dt[\exp\{\mathbf{A}(t - t_0)\}] \\ &\quad = \mathbf{A} + \mathbf{A}^2(t - t_0) + \mathbf{A}^3(t - t_0)^2/2! + \mathbf{A}^4(t - t_0)^3/3! + \cdots + \mathbf{A}^{k+1}(t - t_0)^k/k! + \cdots \\ &\quad = \mathbf{A}[\mathbf{I} + \mathbf{A}(t - t_0) + \mathbf{A}^2(t - t_0)^2/2! + \mathbf{A}^3(t - t_0)^3/3! + \cdots + \mathbf{A}^k(t - t_0)^k/k! + \cdots] \\ &\quad = \mathbf{A}\exp\{\mathbf{A}(t - t_0)\}\end{aligned} \tag{4.20}$$

(Note that the right-hand side of Eq. (4.20) can also be expressed as $[\mathbf{I} + \mathbf{A}(t - t_0) + \mathbf{A}^2(t - t_0)^2/2! + \mathbf{A}^3(t - t_0)^3/3! + \cdots + \mathbf{A}^k(t - t_0)^k/k! + \cdots]\mathbf{A} = \exp\{\mathbf{A}(t - t_0)\}\mathbf{A}$, which implies that $\mathbf{A}\exp\{\mathbf{A}(t - t_0)\} = \exp\{\mathbf{A}(t - t_0)\}\mathbf{A}$.) Substituting Eq. (4.20) into Eq. (4.19), and using Eq. (4.16), we get

$$\mathbf{x}^{(1)}(t) = \mathbf{A}\exp\{\mathbf{A}(t - t_0)\}\mathbf{x}_0 = \mathbf{A}\mathbf{x}(t) \tag{4.21}$$

which is the same as Eq. (4.15). Hence, our definition of the matrix exponential by Eq. (4.18) does satisfy Eq. (4.15). In Eq. (4.20) we saw that the matrix exponential, $\exp\{\mathbf{A}(t-t_0)\}$, *commutes* with the matrix, $\mathbf{A}$, i.e. $\mathbf{A}\exp\{\mathbf{A}(t-t_0)\} = \exp\{\mathbf{A}(t-t_0)\}\mathbf{A}$. This is a special property of $\exp\{\mathbf{A}(t-t_0)\}$, because only rarely do two matrices commute with one another (see Appendix B). Looking at Eq. (4.16), we see that the matrix exponential, $\exp\{\mathbf{A}(t-t_0)\}$, performs a *linear transformation* on the initial state-vector, $\mathbf{x}(t_0)$, to give the state-vector at time t, $\mathbf{x}(t)$. Hence, $\exp\{\mathbf{A}(t-t_0)\}$, is also known as the *state-transition matrix*, as it *transitions* the system given by the homogeneous state-equation, Eq. (4.15), from the state, $\mathbf{x}(t_0)$, at time, t_0, to the state $\mathbf{x}(t)$, at time, t. Thus, using the state-transition matrix we can find the state at any time, t, if we know the state at any previous time, $t_0 < t$. Table 4.1 shows some important properties of the state-transition matrix, which you can easily verify from the definition of $\exp\{\mathbf{A}(t-t_0)\}$, Eq. (4.18).

Now that we know how to solve for the initial response (i.e. response when $\mathbf{u}(t) = \mathbf{0}$) of the system given by Eq. (4.8), let us try to obtain the general solution, $\mathbf{x}(t)$, when the input vector, $\mathbf{u}(t)$, is non-zero for $t \geq t_0$. Again, we will use the steps similar to those for the scalar state-equation, i.e. Eqs. (4.1)–(4.5). However, since now we are dealing with matrix equation, we have to be careful with the sequence of matrix multiplications. Pre-multiplying Eq. (4.8) by $\exp\{-\mathbf{A}(t-t_0)\}$, we get

$$\exp\{-\mathbf{A}(t-t_0)\}\mathbf{x}^{(1)}(t) = \exp\{-\mathbf{A}(t-t_0)\}\mathbf{A}\mathbf{x}(t) + \exp\{-\mathbf{A}(t-t_0)\}\mathbf{B}\mathbf{u}(t) \quad (4.22)$$

Bringing the terms involving $\mathbf{x}(t)$ to the left-hand side, we can write

$$\exp\{-\mathbf{A}(t-t_0)\}[\mathbf{x}^{(1)}(t) - \mathbf{A}\mathbf{x}(t)] = \exp\{-\mathbf{A}(t-t_0)\}\mathbf{B}\mathbf{u}(t) \quad (4.23)$$

From Table 4.1 we note that $d/dt[\exp\{-\mathbf{A}(t-t_0)\}] = -\exp\{-\mathbf{A}(t-t_0)\}\mathbf{A}$. Therefore, the left-hand side of Eq. (4.23) can be expressed as follows:

$$\begin{aligned}
&\exp\{-\mathbf{A}(t-t_0)\}\mathbf{x}^{(1)}(t) - \exp\{-\mathbf{A}(t-t_0)\}\mathbf{A}\mathbf{x}(t) \\
&\quad = \exp\{-\mathbf{A}(t-t_0)\}\mathbf{x}^{(1)}(t) + d/dt[\exp\{-\mathbf{A}(t-t_0)\}]\mathbf{x}(t) \\
&\quad = d/dt[\exp\{-\mathbf{A}(t-t_0)\}\mathbf{x}(t)] \qquad (4.24)
\end{aligned}$$

Hence, Eq. (4.23) can be written as

$$d/dt[\exp\{-\mathbf{A}(t-t_0)\}\mathbf{x}(t)] = \exp\{-\mathbf{A}(t-t_0)\}\mathbf{B}\mathbf{u}(t) \quad (4.25)$$

Table 4.1 Some important properties of the state-transition matrix

S. No.	Property	Expression
1	Stationarity	$\exp\{\mathbf{A}(t_0-t_0)\} = \mathbf{I}$
2	Commutation with $\mathbf{A}$	$\mathbf{A}\exp\{\mathbf{A}(t-t_0)\} = \exp\{\mathbf{A}(t-t_0)\}\mathbf{A}$
3	Differentiation with time, t	$d/dt[\exp\{\mathbf{A}(t-t_0)\}] = \exp\{\mathbf{A}(t-t_0)\}\mathbf{A}$
4	Inverse	$[\exp\{\mathbf{A}(t-t_0)\}]^{-1} = \exp\{\mathbf{A}(t_0-t)\}$
5	Time-marching	$\exp\{\mathbf{A}(t-t_1)\}\exp\{\mathbf{A}(t_1-t_0)\} = \exp\{\mathbf{A}(t-t_0)\}$