Integrating Eq. (4.25) with respect to time, from t_0 to t, we get

$$\exp\{-\mathbf{A}(t-t_0)\}\mathbf{x}(t) - \mathbf{x}(t_0) = \int_{t_0}^{t} \exp\{-\mathbf{A}(\tau - t_0)\}\mathbf{B}\mathbf{u}(\tau)d\tau \tag{4.26}$$

Pre-multiplying both sides of Eq. (4.26) by $\exp\{\mathbf{A}(t-t_0)\}$, and noting from Table 4.1 that $\exp\{-\mathbf{A}(t-t_0)\} = [\exp\{\mathbf{A}(t-t_0)\}]^{-1}$, we can write the solution state-vector as follows:

$$\mathbf{x}(t) = \exp\{\mathbf{A}(t-t_0)\}\mathbf{x}(t_0) + \int_{t_0}^{t} \exp\{\mathbf{A}(t-\tau)\}\mathbf{B}\mathbf{u}(\tau)d\tau; \quad (t \geq t_0) \tag{4.27}$$

Note that the matrix equation, Eq. (4.27), is of the same form as the scalar equation, Eq. (4.5). Using Eq. (4.27), we can calculate the solution to the general matrix state-equation, Eq. (4.8), for $t \geq t_0$. However, we do not yet know how to calculate the state-transition matrix, $\exp\{\mathbf{A}(t-t_0)\}$.

4.2 Calculation of the State-Transition Matrix

If we can calculate the state-transition matrix, $\exp\{\mathbf{A}(t-t_0)\}$, when the state-dynamics matrix, $\mathbf{A}$, and the times, t_0 and $t \geq t_0$, are specified, our task of solving the linear state-equations will simply consist of plugging $\exp\{\mathbf{A}(t-t_0)\}$ into Eq. (4.27) and getting the solution $\mathbf{x}(t)$, provided we know the initial state-vector, $\mathbf{x}(t_0)$, and the input vector, $\mathbf{u}(t)$, for $t \geq t_0$. As stated at the beginning of Section 4.1, the easiest way to solve a matrix state-equation is by decoupling the individual scalar state-equations, which is possible only if the system has distinct eigenvalues. First, let us calculate the state-transition matrix for such a system.

For a linear system of order n, having *n distinct eigenvalues*, $\lambda_1, \lambda_2, \ldots, \lambda_n$, the *eigenvalue problem* (see Chapter 3) is written as follows:

$$\mathbf{A}\mathbf{v}_k = \lambda_k \mathbf{v}_k; \quad (k = 1, 2, \ldots, n) \tag{4.28}$$

We know from Chapter 3 that such a system can be decoupled (or *diagonalized*) by using the following state-transformation:

$$\mathbf{x}'(t) = \mathbf{T}\mathbf{x}(t); \quad \mathbf{T} = [\mathbf{v}_1; \mathbf{v}_2; \ldots; \mathbf{v}_n]^{-1} \tag{4.29}$$

and the state-dynamics matrix then becomes diagonalized as follows:

$$\mathbf{A}' = \mathbf{T}\mathbf{A}\mathbf{T}^{-1} = \begin{bmatrix} \lambda_1 & 0 & 0 & \ldots & 0 \\ 0 & \lambda_2 & 0 & \ldots & 0 \\ . & . & . & \ldots & . \\ 0 & 0 & 0 & \ldots & \lambda_n \end{bmatrix} \tag{4.30}$$

You can easily show from the definition of the state-transition matrix, Eq. (4.18), that the state-transition matrix for the decoupled system is given by

$$\exp\{\mathbf{A}(t-t_0)\} = \begin{bmatrix} \exp\{\lambda_1(t-t_0)\} & 0 & 0 & \dots & 0 \\ 0 & \exp\{\lambda_2(t-t_0)\} & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & 0 & \dots & \exp\{\lambda_n(t-t_0)\} \end{bmatrix} \tag{4.31}$$

Equation (4.31) shows that the state-transition matrix for a decoupled system is a diagonal matrix. In general, the state-transition matrix for any transformed system, $\mathbf{x}'(t) = \mathbf{Tx}(t)$, can be expressed as

$$\begin{aligned}
&\exp\{\mathbf{A}(t-t_0)\} \\
&\quad = \mathbf{I} + \mathbf{A}'(t-t_0) + (\mathbf{A}')^2(t-t_0)^2/2! + (\mathbf{A}')^3(t-t_0)^3/3! + \cdots + (\mathbf{A}')^k(t-t_0)^k/k! + \cdots \\
&\quad = \mathbf{TT}^{-1} + (\mathbf{TAT}^{-1})(t-t_0) + (\mathbf{TAT}^{-1})^2(t-t_0)^2/2! \\
&\qquad + (\mathbf{TAT}^{-1})^3(t-t_0)^3/3! + \cdots + (\mathbf{TAT}^{-1})^k(t-t_0)^k/k! + \cdots \\
&\quad = \mathbf{T}[\mathbf{I} + \mathbf{A}(t-t_0) + \mathbf{A}^2(t-t_0)^2/2! + \mathbf{A}^3(t-t_0)^3/3! + \cdots + \mathbf{A}^k(t-t_0)^k/k! + \cdots]\mathbf{T}^{-1} \\
&\quad = \mathbf{T}\exp\{\mathbf{A}(t-t_0)\}\mathbf{T}^{-1}
\end{aligned} \tag{4.32}$$

Example 4.2

Let us calculate the state-transition matrix of the following system, and then solve for the state-vector if the initial condition is $\mathbf{x}(0) = [\,1;\quad 0\,]^T$ and the applied input is $u(t) = 0$:

$$\mathbf{A} = \begin{bmatrix} -1 & 2 \\ -1 & -3 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \tag{4.33}$$

The eigenvalues of the system are obtained by solving the following characteristic equation:

$$|\lambda\mathbf{I} - \mathbf{A}| = \begin{vmatrix} (\lambda+1) & -2 \\ 1 & (\lambda+3) \end{vmatrix} = (\lambda+1)(\lambda+3) + 2 = \lambda^2 + 4\lambda + 5 = 0 \tag{4.34}$$

which gives the following eigenvalues:

$$\lambda_{1,2} = -2 \pm i \tag{4.35}$$

Note that the negative real parts of both the eigenvalues indicate an asymptotically stable system. Since the eigenvalues are distinct, the system can be decoupled using the state-transformation given by Eq. (4.29). The eigenvectors, $\mathbf{v}_1 = [v_{11};\ v_{21}]^T$ and $\mathbf{v}_2 = [v_{12};\ v_{22}]^T$ are calculated from Eq. (4.28). The equation $\mathbf{Av}_1 = \lambda_1\mathbf{v}_1$ yields the following scalar equations:

$$\lambda_1 v_{11} = -v_{11} + 2v_{21} \tag{4.36a}$$

$$\lambda_1 v_{21} = -v_{11} - 3v_{21} \tag{4.36b}$$

Note that Eqs. (4.36a) and (4.36b) are *linearly dependent*, i.e. we cannot get the *two unknowns*, v_{11}, and v_{21}, by solving these two equations. You may verify this fact by trying to solve for v_{11} and v_{21}. (This behavior of the eigenvector equations is true for a general system of order n; only $(n-1)$ equations relating the eigenvector elements are linearly independent). The best we can do is *arbitrarily specify* one of the two unknowns, and use *either* Eq. (4.36a) *or* Eq. (4.36b) – since both give us the same relationship between v_{11} and v_{21} – to get the remaining unknown. Let us arbitrarily choose $v_{11} = 1$. Then either Eq. (4.36a) or (4.36b) gives us $v_{21} = (1+\lambda_1)/2 = (-1+i)/2$. Hence, the first eigenvector is $\mathbf{v}_1 = [\,1;\quad (-1+i)/2\,]^T$. Similarly, the second eigenvector is obtained by 'solving' $\mathbf{A}\mathbf{v}_2 = \lambda_2\mathbf{v}_2$, yielding the second eigenvector as $\mathbf{v}_2 = [\,1;\quad (-1-i)/2\,]^T$. Plugging the two eigenvectors in Eq. (4.29), we get the state-transformation matrix, $\mathbf{T}$, as

$$\mathbf{T} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 \\ (-1+i)/2 & (-1-i)/2 \end{bmatrix}^{-1} = \begin{bmatrix} (1-i)/2 & -i \\ (1+i)/2 & i \end{bmatrix} \tag{4.37}$$

Then the diagonalized state-dynamics matrix, $\mathbf{A}'$, is given by

$$\mathbf{A}' = \mathbf{TAT}^{-1} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} (-2+i) & 0 \\ 0 & (-2-i) \end{bmatrix} \tag{4.38}$$

and the state-transition matrix for the transformed system is

$$\mathrm{e}^{\mathbf{A}'t} = \begin{bmatrix} \exp(\lambda_1 t) & 0 \\ 0 & \exp(\lambda_2 t) \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{(-2+i)t} & 0 \\ 0 & \mathrm{e}^{(-2-i)t} \end{bmatrix} \tag{4.39}$$

Note that $t_0 = 0$ in this example. Then from Eq. (4.32) the state-transition matrix for the original system is given by

$$\begin{aligned} \mathrm{e}^{\mathbf{A}t} &= \mathbf{T}^{-1}\mathrm{e}^{\mathbf{A}'t}\mathbf{T} \\ &= \begin{bmatrix} 1 & 1 \\ (-1+i)/2 & (-1-i)/2 \end{bmatrix} \begin{bmatrix} \mathrm{e}^{(-2+i)t} & 0 \\ 0 & \mathrm{e}^{(-2-i)t} \end{bmatrix} \begin{bmatrix} (1-i)/2 & -i \\ (1+i)/2 & i \end{bmatrix} \\ &= \begin{bmatrix} [(1-i)\mathrm{e}^{(-2+i)t} + (1+i)\mathrm{e}^{(-2-i)t}]/2 & i(\mathrm{e}^{(-2-i)t} - \mathrm{e}^{(-2+i)t}) \\ i(\mathrm{e}^{(-2+i)t} - \mathrm{e}^{(-2-i)t})/2 & [(1+i)\mathrm{e}^{(-2+i)t} + (1-i)\mathrm{e}^{(-2-i)t}]/2 \end{bmatrix} \end{aligned} \tag{4.40}$$

Those with a taste for complex algebra may further simplify Eq. (4.40) by using the identity $\mathrm{e}^{a+ib} = \mathrm{e}^a[\cos(b) + i\sin(b)]$, where a and b are real numbers. The resulting expression for $\mathrm{e}^{\mathbf{A}t}$ is as follows:

$$\mathrm{e}^{\mathbf{A}t} = \begin{bmatrix} \mathrm{e}^{-2t}[\cos(t) + \sin(t)] & 2\mathrm{e}^{-2t}\sin(t) \\ -\mathrm{e}^{-2t}\sin(t) & \mathrm{e}^{-2t}[\cos(t) - \sin(t)] \end{bmatrix}; \quad (t \geq 0) \tag{4.41}$$

The solution, $\mathbf{x}(t)$, is then given by Eq. (4.27) with $\mathbf{u}(t) = 0$ as follows:

$$\mathbf{x}(t) = [x_1(t); x_2(t)]^T = \mathrm{e}^{\mathbf{A}t}\mathbf{x}(0) = \begin{bmatrix} \mathrm{e}^{-2t}[\cos(t) + \sin(t)] \\ -\mathrm{e}^{-2t}sin(t) \end{bmatrix}; \quad (t \geq 0) \tag{4.42}$$

The state variables, $x_1(t)$ and $x_2(t)$, given by Eq. (4.42) are plotted in Figure 4.1. Note that both the state variables shown in Figure 4.1 decay to zero in about 3 s, thereby confirming that the system is asymptotically stable.

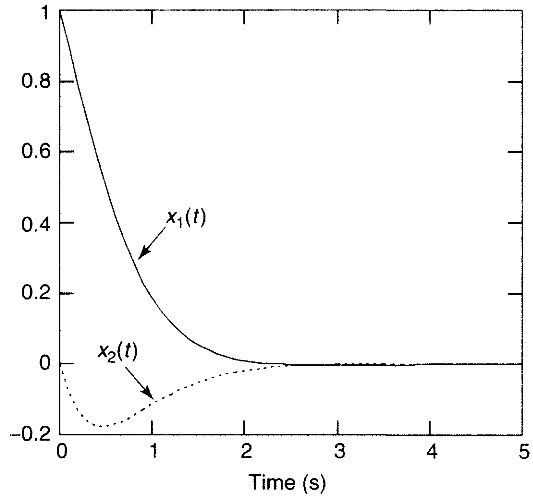

Figure 4.1 The calculated state variables, $x_1(t)$ and $x_2(t)$, for Example 4.2

The method presented in Example 4.2 for calculating the state-transition matrix is restricted to those systems which have distinct eigenvalues. The intrinsic MATLAB function *expm3* lets you use the diagonalization method for the calculation of the matrix exponential for systems with distinct eigenvalues as follows:

```
>>eP = expm3(P) <enter>
```

where **P** is a square matrix of which the matrix exponential, **eP**, is to be calculated. Alternatively, you can use the intrinsic MATLAB function *eig* as follows to calculate the eigenvector matrix, **V**, and the diagonalized matrix, **D**, with eigenvalues of **P** as its diagonal elements:

```
>>[V,D] = eig(P) <enter>
```

Then use the MATLAB function *exp* as follows to calculate the matrix exponential of **D**:

```
>>eD = exp(D) - rot90(eye(size(D))) <enter>
```

The MATLAB function *exp(D)* calculates a matrix whose elements are exponentials of the corresponding elements of the matrix **D**. The matrix exponential of **D** is obtained by subtracting the *off-diagonal* elements of *exp(D)* (which are all ones) from *exp(D);* this is done by forming a matrix whose diagonal elements are *zeros* and whose off-diagonal elements are all *ones*–an *identity matrix* of same size as **D**, *eye(size(D)), rotated by 90* *degrees* using the command *rot90(eye(size(D)))*. Finally, the matrix exponential of **P** can be obtained as follows, using $\mathbf{eP} = \mathbf{VeDV}^{-1}$:

```
>>eP = V*eD*inv(V) <enter>
```

Example 4.3

Using MATLAB, let us calculate the state-transition matrix for the system in Example 4.2 for $t_0 = 0$ and $t = 2$ s. First, let us use the command *expm3* as follows:

```
>>A=[-1 2; -1 -3]; eAt = expm3(A*2) <enter>

eAt =
9.0324e-003    3.3309e-002
-1.6654e-002   -2.4276e-002
```

Now let us use the alternative approach with the command *eig* as follows:

```
>>[V, D] = eig(A*2) <enter>

V =

8.1650e-001                     8.1650e-001
-4.0825e-001 +4.0825e-001i   -4.0825e-001 -4.0825e-001i

D =

-4.0000e+000+ 2.0000e+000i      0

      0        -4.0000e+000- 2.0000e+000i
```

Then the state-transition matrix of the diagonalized system is calculated as follows:

```
>>eD = exp(D) - rot90(eye(size(D))) <enter>

eD =
-7.6220e-003 +1.6654e-002i   0
      0                     -7.6220e-003 -1.6654e-002i
```

Finally, using the inverse state-transformation from the diagonalized system to the original system, we get the state-transition matrix, $e^{\mathbf{A}t}$, as follows:

```
>>eAt = V*eD*inv(V) <enter>

eAt =

9.0324e-003    3.3309e-002    -2.2646e-018i
-1.6654e-002   -2.4276e-002   +1.6505e-018i
```

which is the same result as that obtained using *expm3* (ignoring the negligible imaginary parts). You may verify the accuracy of the computed value of $e^{\mathbf{A}t}$ by comparing it with the exact result obtained in Eq. (4.41) for $t = 2$ s.

For systems with repeated eigenvalues, a general method of calculating the state-transition matrix is the Laplace transform method, in which the Laplace transform is taken of the homogeneous state-equation, Eq. (4.15) subject to the initial condition, $\mathbf{x}(0) = \mathbf{x}_0$ as follows:

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) \tag{4.43}$$

where $\mathbf{X}(s) = \mathcal{L}[\mathbf{x}(t)]$. Collecting the terms involving $\mathbf{X}(s)$ to the left-hand side of Eq. (4.43), we get

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{x}(0) \tag{4.44}$$

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}_0 \tag{4.45}$$

Taking the inverse Laplace transform of Eq. (4.45), we get the state-vector, $\mathbf{x}(t)$, as

$$\mathbf{x}(t) = \mathcal{L}^{-1}[\mathbf{X}(s)] = \mathcal{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}_0] = \mathcal{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}]\mathbf{x}_0 \tag{4.46}$$

Comparing Eq. (4.46) with Eq. (4.16), we obtain the following expression for the state-transition matrix:

$$e^{\mathbf{A}t} = \mathcal{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}] \tag{4.47}$$

Thus, Eq. (4.47) gives us a general method for calculating the state-transition matrix for $t > 0$. The matrix $(s\mathbf{I} - \mathbf{A})^{-1}$ is called the *resolvent* because it helps us in solving the state-equation by calculating $e^{\mathbf{A}t}$. If the initial condition is specified at $t = t_0$, we would be interested in the state-transition matrix, $\exp\{\mathbf{A}(t - t_0)\}$, for $t > t_0$, which is obtained from Eq. (4.47) merely by substituting t by $(t - t_0)$.

Example 4.4

Consider a system with the following state-dynamics matrix:

$$\mathbf{A} = \begin{bmatrix} -2 & 1 & 5 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix} \tag{4.48}$$

Let us calculate the state-transition matrix and the initial response, if the initial condition is $\mathbf{x}(0) = [0; \quad 0; \quad 1]^T$. The eigenvalues of the system are calculated as follows:

$$|\lambda\mathbf{I} - \mathbf{A}| = \begin{vmatrix} (\lambda + 2) & -1 & -5 \\ 0 & \lambda & 3 \\ 0 & 0 & \lambda \end{vmatrix} = \lambda^2(\lambda + 2) = 0 \tag{4.49}$$

From Eq. (4.49) it follows that the eigenvalues of the system are $\lambda_1 = \lambda_2 = 0$, and $\lambda_3 = -2$. Since the first two eigenvalues are repeated, the system cannot be decoupled, and the approach of Example 4.2 for calculating the state-transition matrix is inapplicable. Let us apply the Laplace transform approach given by Eq. (4.47). First, the *resolvent* $(s\mathbf{I} - \mathbf{A})^{-1}$ is calculated as follows:

$$(s\mathbf{I}-\mathbf{A})^{-1} = \mathrm{adj}((s\mathbf{I}-\mathbf{A})/|s\mathbf{I}-\mathbf{A}| = 1/[s^2(s+2)]$$
$$\times \begin{bmatrix} s^2 & 0 & 0 \\ s & s(s+2) & 0 \\ (5s+3) & -3(s+2) & s(s+2) \end{bmatrix}^T$$
$$= \begin{bmatrix} 1/(s+2) & 1/[s(s+2)] & (5s+3)/[s^2(s+2)] \\ 0 & 1/s & -3/s^2 \\ 0 & 0 & 1/s \end{bmatrix} \quad (4.50)$$

Taking the inverse Laplace transform of Eq. (4.50) with the help of partial fraction expansions for the elements of $(s\mathbf{I}-\mathbf{A})^{-1}$ and using Table 2.1, we get the state-transition matrix as follows:

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{-2t} & (1-e^{-2t})/2 & 7(1-e^{-2t})/4 + 3t/2 \\ 0 & 1 & -3t \\ 0 & 0 & 1 \end{bmatrix}; \quad (t>0) \quad (4.51)$$

Note that the inverse Laplace transform of $1/s$ is $u_s(t)$ from Table 2.1. However, since we are interested in finding the state-transition matrix and the response only for $t > 0$ (because the response at $t = 0$ is known from the initial condition, $\mathbf{x}(0)$) we can write $\mathcal{L}^{-1}(1/s) = 1$ for $t > 0$, which has been used in Eq. (4.51). The initial response is then calculated as follows:

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = e^{\mathbf{A}t}\mathbf{x}(0) = \begin{bmatrix} 7(1-e^{-2t})/4 + 3t/2 \\ -3t \\ 1 \end{bmatrix}; \quad (t>0) \quad (4.52)$$

Note that the term $3t/2$ makes $x_1(t)$ keep on increasing with time, $t > 0$. Similarly, $x_2(t)$ keeps on increasing with time. This confirms that the system is unstable. A plot of $x_1(t)$, $x_2(t)$, and $x_3(t)$ is shown in Figure 4.2.

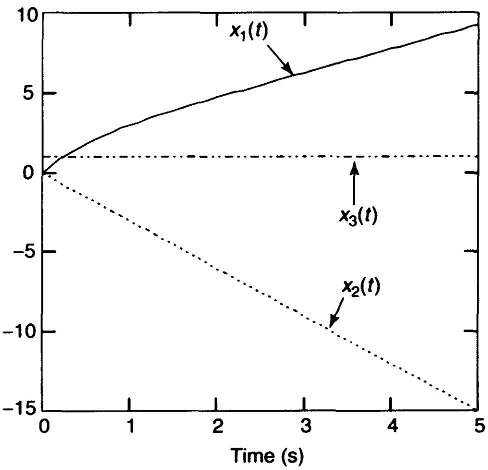

Figure 4.2 The calculated state variables, $x_1(t)$, $x_2(t)$, and $x_3(t)$, for Example 4.4