4.3 Understanding the Stability Criteria through the State-Transition Matrix

In Section 2.6 we listed the criteria by which we can judge whether a system is stable. The state-transition matrix allows us to understand the stability criteria. We saw in Example 4.2 that the elements of $e^{\mathbf{A}t}$ are *linear combinations* of $\exp(\lambda_k t)$, where λ_k for $k = 1, 2, \ldots, n$ are the *distinct* eigenvalues of the system (Eq. (4.40)). Such elements can be expressed as $\exp(a_k t)$ multiplied by oscillatory terms, $\sin(b_k t)$ and $\cos(b_k t)$, where a_k and b_k are real and imaginary parts of the kth eigenvalue, $\lambda_k = a_k + ib_k$ (Eq. (4.41)). If the real parts, a_k, of all the eigenvalues are *negative* – as in Example 4.2 – the initial responses of *all* the state-variables will decay to zero as time, t, becomes large due to the presence of $\exp(a_k t)$ as the factor in all the elements of $e^{\mathbf{A}t}$. Hence, a system with *all* eigenvalues having negative real parts is *asymptotically stable*. This is the first stability criterion. By the same token, if any eigenvalue λ_k, has a *positive* real part, a_k, then the corresponding factor $\exp(a_k t)$ will diverge to infinity as time, t, becomes large, signifying an *unstable* system. This is the second stability criterion.

In Example 4.4, we saw that if a zero eigenvalue is repeated twice, it leads to the presence of terms such as ct, where c is a constant, in the elements of $e^{\mathbf{A}t}$ (Eq. (4.51)). More generally, if an eigenvalue, λ_k, which is repeated *twice* has *zero real part* (i.e. $\lambda_k = ib_k$), then $e^{\mathbf{A}t}$ will have terms such as $t\sin(b_k t)$ and $t\cos(b_k t)$ – and their combinations – in its elements. If an eigenvalue with zero real part is repeated *thrice*, then $e^{\mathbf{A}t}$ will have combinations of $t^2\sin(b_k t)$, $t^2\cos(b_k t)$, $t\sin(b_k t)$, and $t\cos(b_k t)$ in its elements. Similarly, for eigenvalues with zero real parts repeated larger number of times, there will be *higher powers* of t present as *coefficients* of the oscillatory terms in the elements of $e^{\mathbf{A}t}$. Hence, if any eigenvalue, λ_k, having zero real part is *repeated two or more times*, the presence of powers of t as coefficients of the oscillatory terms, $\sin(b_k t)$ and $\cos(b_k t)$, causes elements of $e^{\mathbf{A}t}$ to blow-up as time, t, increases, thereby indicating an *unstable* system. This is the third stability criterion.

Note that individual initial responses to a *specific initial condition* may not be sufficient to tell us whether a system is stable. This is seen in the following example.

Example 4.5

Reconsider the system of Example 4.4 with the initial condition, $\mathbf{x}(0) = [\,1;\quad 0;\quad 0\,]^T$. Substituting the initial condition into Eq. (4.16), we get the following initial response for the system:

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = e^{\mathbf{A}t} \quad \mathbf{x}(0) = \begin{bmatrix} e^{-2t} \\ 0 \\ 0 \end{bmatrix}; \quad (t > 0) \tag{4.53}$$

Equation (4.53) indicates that $\mathbf{x}(t) \to 0$ as $t \to \infty$. Thus, a system we *know* to be *unstable* from Example 4.4 (and from the third stability criterion), has an initial response decaying asymptotically to zero when the initial condition is $\mathbf{x}(0) = [\,1;\quad 0;\quad 0\,]^T$, which is the characteristic of an asymptotically stable system.

Example 4.5 illustrates that we can be fooled into believing that a system is stable if we look at its initial response to only *some specific* initial conditions. A true mirror of the