where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -K/3 & (-2-K/3) & -K/3 & -1/3 \\ K/2 & K/2 & (-3+K/2) & 1/2 \\ -K & -K & -K & 0 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 0 \\ K/3 \\ -K/2 \\ K \end{bmatrix}$$

$$\mathbf{C} = [1 \quad 1 \quad 1 \quad 0]; \quad \mathbf{D} = 0 \tag{5.9}$$

The closed-loop system is of fourth order, as expected. The closed-loop poles are the eigenvalues of **A**, i.e. the solutions of the following characteristic equation:

$$|\lambda\mathbf{I} - \mathbf{A}| = \begin{vmatrix} (\lambda-1) & 0 & 0 & 0 \\ K/3 & (\lambda+2+K/3) & K/3 & 1/3 \\ -K/2 & -K/2 & (\lambda+3-K/2) & -1/2 \\ K & K & K & \lambda \end{vmatrix} = 0 \tag{5.10}$$

It is evident from Eq. (5.10) that, irrespective of the value of K, one of the eigenvalues of **A** is $\lambda = 1$, which corresponds to a closed-loop pole at $s = 1$. Hence, irrespective of the design parameter, K, we have an *unstable* closed-loop system, which means that the chosen design approach of cancelling an unstable pole with a zero *does not work*. More importantly, even though we have an *unconditionally* unstable closed-loop system, the closed-loop transfer function given by Eq. (5.4) *fools us into believing* that we can stabilize the closed-loop system by selecting an *appropriate value* for K. Such a system which remains unstable irrespective of the values of the control design parameters is called an *unstabilizable system*. The classical design approach of Example 5.1 gave us an unstabilizable closed-loop system, and we didn't even know it! Stabilizability of a system is a consequence of an important property known as *controllability*, which we will consider next. (Although we considered a closed-loop system in Example 5.2, the properties *controllability* and *stabilizability* are more appropriately defined for a plant.)

5.2 Controllability

When as children we sat in the back seat of a car, our collective effort to move the car by pushing on the front seat always ended in failure. This was because the input we provided to the car in this manner, *no matter how large*, did not affect the overall motion of the car. There was something known as the *third law of Newton*, which physically prevented us from achieving our goal. Hence, for us the car was *uncontrollable when we were sitting in the car*. The same car could be moved, however, by stepping out and giving a hefty push to it from the outside; then it became a *controllable* system for our purposes. *Controllability* can be defined as the property of a system when it is possible to take the system from *any initial state*, $\mathbf{x}(t_0)$, to *any final state*, $\mathbf{x}(t_f)$, in a *finite* time, $(t_f - t_0)$, by means of the input vector, $\mathbf{u}(t)$, $t_0 \le t \le t_f$. It is important to stress the words *any* and *finite*, because it may be possible to move an uncontrollable system from *some initial* *states* to *some final states*, or take an *infinite* amount of time in moving the uncontrollable system, using the input vector, $\mathbf{u}(t)$. Controllability of a system can be easily determined if we can *decouple* the state-equations of a system. Each decoupled scalar state-equation corresponds to a *sub-system*. If any of the decoupled state-equations of the system is *unaffected* by the input vector, then it is not possible to change the corresponding state variable using the input, and hence, the sub-system is *uncontrollable*. If any sub-system is uncontrollable, i.e. if any of the state variables is unaffected by the input vector, then it follows that the entire system is uncontrollable.

Example 5.3

Re-consider the closed-loop system of Example 5.2. The state-equations of the closed-loop system (Eqs. (5.7)–(5.9)) can be expressed in scalar form as follows:

$$x_1^{(1)}(t) = x_1(t) \tag{5.11a}$$

$$x_2^{(1)}(t) = -Kx_1(t)/3 - (2 + K/3)x_2(t) - Kx_3(t)/3 - x_4(t)/3 + Ky_d(t)/3 \tag{5.11b}$$

$$x_3^{(1)}(t) = Kx_1(t)/2 + Kx_2(t)/2 + (-3 + K/2)x_3(t) + x_4(t)/2 - Ky_d(t)/2 \tag{5.11c}$$

$$x_4^{(1)}(t) = -Kx_1(t) - Kx_2(t) - Kx_3(t) + Ky_d(t) \tag{5.11d}$$

On examining Eq. (5.11a), we find that the equation is decoupled from the other state-equations, and does not contain the input to the closed-loop system, $y_d(t)$. Hence, the state variable, $x_1(t)$, is entirely unaffected by the input, $y_d(t)$, which implies that the system is *uncontrollable*. Since the uncontrollable sub-system described by Eq. (5.11a) is also unstable (it corresponds to the eigenvalue $\lambda = 1$), there is no way we can stabilize the closed-loop system by changing the controller design parameter, K. Hence, the system is *unstabilizable*. In fact, the plant of this system given by the state-space representation of Eq. (5.5) is itself unstabilizable, because of the zero in the matrix $\mathbf{B_p}$ corresponding to the sub-system having eigenvalue $\lambda = 1$. The unstabilizable plant leads to an unstabilizable closed-loop system.

Example 5.3 shows how a decoupled state-equation indicating an *uncontrollable* and *unstable* sub-system implies an *unstabilizable* system.

Example 5.4

Let us analyze the controllability of the following system:

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \tag{5.12}$$

The system is *unstable*, with four zero eigenvalues. Since the state-equations of the system are coupled, we cannot directly deduce controllability. However, some of the state-equations can be decoupled by transforming the state-equations using the transformation $\mathbf{z}(t) = \mathbf{T}\mathbf{x}(t)$, where

$$\mathbf{T} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \tag{5.13}$$

The transformed state-equations can be written in the following scalar form:

$$z_1^{(1)}(t) = x_3'(t) \tag{5.14a}$$

$$z_2^{(1)}(t) = x_4'(t) \tag{5.14b}$$

$$z_3^{(1)}(t) = 0 \tag{5.14c}$$

$$z_4^{(1)}(t) = -2u(t) \tag{5.14d}$$

Note that the state-equation, Eq. (5.14c) denotes an uncontrollable sub-system in which the state variable, $z_3(t)$, is unaffected by the input, $u(t)$. Hence, the system is uncontrollable. However, since the only uncontrollable sub-system denoted by Eq. (5.14c) is *stable* (its eigenvalue is, $\lambda = 0$), we can safely *ignore* this sub-system and *stabilize* the remaining sub-systems denoted by Eqs. (5.14a),(5.14b), and (5.14d), using a feedback controller that modifies the control input, $u(t)$. An uncontrollable system all of whose uncontrollable sub-systems are stable is thus said to be *stabilizable*. The process of stabilizing a stabilizable system consists of ignoring all uncontrollable but stable sub-systems, and designing a controller based on the remaining (controllable) sub-systems. Such a control system will be successful, because each ignored sub-system will be stable.

In the previous two examples, we could determine controllability, only because certain state-equations were decoupled from the other state-equations. Since decoupling state-equations is a cumbersome process, and may not be always possible, we need another criterion for testing whether a system is controllable. The following *algebraic controllability test theorem* provides an easy way to check for controllability.

Theorem

A linear, time-invariant system described by the matrix state-equation, $\mathbf{x}^{(1)}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$ *is* controllable *if and only if the* controllability test matrix

$$\mathbf{P} = [\mathbf{B};\quad \mathbf{AB};\quad \mathbf{A}^2\mathbf{B};\quad \mathbf{A}^3\mathbf{B};\quad \ldots;\quad \mathbf{A}^{n-1}\mathbf{B}]$$

is of rank *n, the order of the system.*

(The *rank* of a matrix, $\mathbf{P}$, is defined as the dimension of the *largest non-zero* determinant formed out of the matrix, $\mathbf{P}$ (see Appendix B). If $\mathbf{P}$ is a square matrix, the largest determinant formed out of **P** is $|\mathbf{P}|$. If **P** is not a square matrix, the largest determinant formed out of **P** is either the determinant formed by taking all the rows and equal number of columns, or all the columns and equal number of rows of **P**. See Appendix B for an illustration of the rank of a matrix. Note that for a system of order n with r inputs, the size of the controllability test matrix, **P**, is $(n \times nr)$. The largest non-zero determinant of **P** can be of dimension n. Hence, the rank of **P** can be *either less than or equal* to n.)

A rigourous proof of the algebraic controllability test theorem can be found in Friedland [2]. An analogous form of algebraic controllability test theorem can be obtained for linear, time-varying systems [2]. Alternatively, we can form a *time-varying* controllability test matrix as

$$\mathbf{P}(t) = [\mathbf{B}(t); \quad \mathbf{A}(t)\mathbf{B}(t); \quad \mathbf{A}^2(t)\mathbf{B}(t); \quad \mathbf{A}^3(t)\mathbf{B}(t); \quad \ldots; \quad \mathbf{A}^{n-1}(t)\mathbf{B}(t)] \tag{5.15}$$

and check the rank of $\mathbf{P}(t)$ for all times, $t \geq t_0$, for a linear, time-varying system. If at any instant, t, the rank of $\mathbf{P}(t)$ is less than n, the system is uncontrollable. However, we must use the time-varying controllability test matrix of Eq. (5.15) with great *caution*, when the state-coefficient matrices are rapidly changing with time, because the test can be practically applied at *discrete time step* – rather than at all possible times (see Chapter 4) – and there may be some time intervals (smaller than the time steps) in which the system may be uncontrollable.

Example 5.5

Using the controllability test theorem, let us find whether the following system is controllable:

$$\mathbf{A} = \begin{bmatrix} -2 & 1 \\ -1 & -3 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{5.16}$$

The controllability test matrix is the following:

$$\mathbf{P} = [\mathbf{B}; \quad \mathbf{AB}] = \begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix} \tag{5.17}$$

The largest determinant of **P** is $|\mathbf{P}| = -1 \neq 0$, Hence the rank of **P** is equal to 2, the order of the system. Thus, by the controllability test theorem, the system is *controllable*.

Applying the algebraic controllability test involves finding the rank of **P**, and checking whether it is equal to n. This involves forming all possible determinants of dimension n out of the matrix **P**, by removing some of the columns (if $m > 1$), and checking whether all of those determinants are non-zero. By any account, such a process is cumbersome if performed by hand. However, MATLAB provides us the command *rank*(P) for finding the rank of a matrix, **P**. Moreover, MATLAB's Control System Toolbox (CST) lets you directly form the controllability test matrix, **P**, using the command *ctrb* as follows:

```
>>P = ctrb(A, B) <enter>
```

or

```
>>P = ctrb(sys) <enter>
```

where **A** and **B** are the state coefficient matrices of the system whose LTI object is *sys*.

Example 5.6

Let us verify the uncontrollability of the system given in Example 5.4, using the controllability test. The controllability test matrix is constructed as follows:

```
>>A=[0 0 1 0; zeros(1,3)1; zeros(2,4)]; B=[0 0 -1 1]'; P=ctrb(A,B)
  <enter>

P =

    0  -1   0   0
    0   1   0   0
   -1   0   0   0
    1   0   0   0
```

Then the rank of **P** is found using the MATLAB command *rank:*

```
>>rank(P) <enter>

ans =

 2
```

Since the rank of **P** is *less than* 4, the order of the system, it follows from the controllability test theorem that the system is *uncontrollable*.

What are the causes of uncontrollability? As our childhood attempt of pushing a car while sitting inside it indicates, whenever we choose an input vector that does not affect *all* the state variables *physically*, we will have an uncontrollable system. An attempt to cancel a pole of the plant by a zero of the controller may also lead to an uncontrollable closed-loop system *even though the plant itself may be controllable*. Whenever you see a system in which pole-zero cancellations have occurred, the chances are high that such a system is uncontrollable.

Example 5.7

Let us analyze the controllability of the closed-loop system of configuration shown in Figure 2.32, in which the controller, $H(s)$, and plant, $G(s)$, are as follows:

$$H(s) = K(s-2)/(s+1); \quad G(s) = 3/(s-2) \tag{5.18}$$

The closed-loop transfer function in which a pole-zero cancellation has occurred at $s = 2$ is the following:

$$Y(s)/Y_d(s) = G(s)H(s)/[1 + G(s)H(s)] = 3K/(s + 3K + 1) \qquad (5.19)$$

The Jordan canonical form of the plant is the following:

$$\mathbf{A_p} = 2; \quad \mathbf{B_p} = 3; \quad \mathbf{C_p} = 1; \quad \mathbf{D_p} = 0 \qquad (5.20)$$

Note that the plant is controllable (the controllability test matrix for the plant is just $\mathbf{P} = \mathbf{B_p}$, which is of rank 1). The Jordan canonical form of the controller is the following:

$$\mathbf{A_c} = -1; \quad \mathbf{B_c} = K; \quad \mathbf{C_c} = -3; \quad \mathbf{D_c} = K \qquad (5.21)$$

The closed-loop state-space representation is obtained using Eqs. (3.146)–(3.148) as the following:

$$\mathbf{A} = \begin{bmatrix} (2-3K) & -9 \\ -K & -1 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 3K \\ K \end{bmatrix}$$
$$\mathbf{C} = [1 \quad 0]; \quad \mathbf{D} = 0 \qquad (5.22)$$

The controllability test matrix for the closed-loop system is the following:

$$\mathbf{P} = [\mathbf{B} \quad \mathbf{AB}] = \begin{bmatrix} 3K & -(9K^2+3K) \\ K & -(3K^2+K) \end{bmatrix} \qquad (5.23)$$

To see whether $\mathbf{P}$ is of rank 2 (i.e. whether $\mathbf{P}$ is non-singular) let us find its determinant as follows:

$$|\mathbf{P}| = \begin{vmatrix} 3K & -(9K^2+3K) \\ K & -(3K^2+K) \end{vmatrix} = -9K^3 - 3K^2 + 9K^3 + 3K^2 = 0 \qquad (5.24)$$

Since $|\mathbf{P}| = 0$, $\mathbf{P}$ is singular, its rank is less than 2. Therefore, the closed-loop system is uncontrollable no matter what value of the controller design parameter, K, is chosen. Hence, a *controllable* plant has led to an *uncontrollable* closed-loop system in which a pole-zero cancellation has occurred.

Other causes of uncontrollability could be *mathematical*, such as using *superfluous* state variables (i.e. more state variables than the order of the system) when modeling a system; the superfluous state variables will be definitely unaffected by the inputs to the system, causing the state-space representation to be uncontrollable, even though the system may be physically controllable. A rare cause of uncontrollability is *too much symmetry* in the system's mathematical model. Electrical networks containing *perfectly balanced bridges* are examples of systems with too much symmetry. However, perfect symmetry almost never exists in the real world, or in its digital computer model.