5.4 Observers, Observability, and Compensators

When we designed control systems using full-state feedback in the previous section, it was assumed that we can measure and feedback all the state variables of the plant using sensors. However, it is rarely possible to measure all the state variables. Some state variables are not even physical quantities. Even in such cases where all the state variables are physical quantities, accurate sensors may not be available, or may be too expensive to construct for measuring all the state variables. Also, some state variable measurements can be so noisy that a control system based on such measurements would be unsuccessful. Hence, it is invariably required to *estimate* rather than measure the state-vector of a system. How can one estimate the state-vector, if it cannot be measured? The answer lies in *observing* the output of the system for a known input and for a finite time interval, and then reconstructing the state-vector from the record of the output. The mathematical model of the process by which a state-vector is estimated from the measured output and the known input is called an *observer* (or *state estimator*). An observer is an essential part of modern control systems. When an observer estimates the *entire* state-vector, it is called a *full-order observer*. However, the state variables that can be measured need not be estimated, and can be directly deduced from the output. An observer which estimates only the unmeasurable state variables is called the *reduced-order observer*. A reduced-order observer results in a smaller order control system, when compared to the full-order observer. However, when the measured state variables are noisy, it is preferable to use a full-order observer to reduce the effect of noise on the control system. A controller which generates the control input to the plant based on the estimated state-vector is called a *compensator*. We will consider the design of observers and compensators below.

Before we can design an observer for a plant, the plant must be *observable*. *Observability* is an important property of a system, and can be defined as the property that makes it possible to determine *any initial state*, $\mathbf{x}(t_0)$, of an *unforced* system (i.e. when the input vector, $\mathbf{u}(t)$, is *zero*) by using a *finite record* of the output, $\mathbf{y}(t)$. The term *finite record* implies that the output is recorded for only a *finite time interval* beginning at $t = t_0$. In other words, observability is a property which enables us to determine what the system was doing at some time, t_0, after measuring its output for a finite time interval beginning at that time. The term *any initial state* is significant in the definition of observability; it may be possible to determine *some initial states* by recording the output, and the system may yet be *unobservable*. Clearly, observability requires that *all* the state variables must contribute to the output of the system, otherwise we cannot reconstruct *all possible* combinations of state variables (i.e. *any* initial state-vector) by measuring the output. The relationship between observability and the output is thus the *dual* of that between controllability and the input. For a system to be controllable, all the state variables must *be affected* by the input; for a system to be observable, all the state variables must *affect* the output. If there are some state variables which *do not* contribute to the output, then the system is *unobservable*. One way of determining observability is by looking at the decoupled state-equations, and the corresponding output equation of a system.

Example 5.15

Consider a system with the following scalar state-equations:

$$\begin{aligned} x_1^{(1)}(t) &= 2x_1(t) + 3u(t) \\ x_2^{(1)}(t) &= -x_2(t) \\ x_3^{(1)}(t) &= 5x_3(t) - u(t) \end{aligned} \tag{5.90}$$

The scalar output equations of the system are the following:

$$\begin{aligned} y_1(t) &= x_1(t) \\ y_2(t) &= 2x_2(t) + x_1(t) + u(t) \end{aligned} \tag{5.91}$$

Equation (5.90) implies that the state variable, $x_3(t)$, is decoupled from the other two state variables, $x_1(t)$ and $x_2(t)$. Also, $x_3(t)$ does not affect either of the two output variables, $y_1(t)$ and $y_2(t)$. Since the state variable $x_3(t)$, does not contribute to the output vector, $\mathbf{y}(t) = [\, y_1(t); \quad y_2(t)\,]^T$, either *directly* or *indirectly* through $x_1(t)$ and $x_2(t)$, it follows that the system is *unobservable.*

As it is not always possible to decouple the state-equations, we need another way of testing for observability. Similar to the algebraic controllability test theorem, there is an *algebraic observability test theorem* for linear, time-invariant systems stated as follows.

Theorem
The unforced system, $\mathbf{x}^{(1)}(t) = \mathbf{A}\mathbf{x}(t)$, $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$, *is observable if and only if the rank of the* observability test matrix, $\mathbf{N} = [\,\mathbf{C}^{\mathbf{T}};\quad \mathbf{A}^{\mathbf{T}}\mathbf{C}^{\mathbf{T}};\quad (\mathbf{A}^{\mathbf{T}})^2\mathbf{C}^{\mathbf{T}};\quad \ldots;\quad (\mathbf{A}^{\mathbf{T}})^{n-1}\mathbf{C}^{\mathbf{T}}\,]$, *is equal to n, the order of the system.*

The proof of this theorem, given in Friedland [2], follows from the definition of observability, and recalling from Chapter 4 that the output of an unforced (homogeneous) linear, time-invariant system is given by $\mathbf{y}(t) = \mathbf{C}\exp\{\mathbf{A}(t - t_0)\}\mathbf{x}(t_0)$, where $\mathbf{x}(t_0)$ is the initial state-vector.

Example 5.16

Let us apply the observability test theorem to the system of Example 5.15. The state coefficient matrices, **A** and **C**, are the following:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{bmatrix}; \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \end{bmatrix} \tag{5.92}$$

The observability test matrix, **N**, is constructed as follows:

$$\mathbf{A}^{\mathbf{T}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{bmatrix}; \quad \mathbf{C}^{T} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}; \quad \mathbf{A}^{\mathbf{T}}\mathbf{C}^{\mathbf{T}} = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}$$

$$(\mathbf{A}^T)^2\mathbf{C}^T = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad (5.93)$$

or

$$\mathbf{N} = \begin{bmatrix} 1 & 2 & 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (5.94)$$

The entire third row of **N** consists of zeros; hence it is impossible to form a (3 × 3) sized, non-zero determinant out of the rows and columns of **N**. Thus rank (**N**) < 3 for this third order system, therefore the system is *unobservable*.

Rather than forming the observability test matrix, **N**, by hand as in Example 5.16, which could be a tedious process for large order systems, we can use the MATLAB (CST) command *ctrb*, noting that **N** is the controllability test matrix in which **A** is replaced by $\mathbf{A}^T$ and **B** is replaced by $\mathbf{C}^T$. Thus, the command

```
>>N = ctrb(A',C') <enter>
```

will give us the observability test matrix.

The reasons for unobservability of a system are pretty much the same as those for uncontrollability, namely the use of superfluous state variables in state-space model, pole-zero cancellation in the system's transfer matrix, too much symmetry, and physical unobservability (i.e. selection of an output vector which is physically unaffected by one or more state variables). If the sub-systems which cause unobservability are *stable*, we can safely ignore those state variables that do not contribute to the output, and design an observer based on the remaining state variables (which would constitute an observable *sub-system*). Thus a stable, unobservable system is said to be *detectable*. If an unobservable sub-system is *unstable*, then the entire system is said to be *undetectable*, because an observer cannot be designed by ignoring the unobservable (and unstable) sub-system. In Example 5.15, the unobservable sub-system corresponding to the decoupled state variable, $x_3(t)$, is unstable (it has a pole at $s = 5$). Hence, the system of Example 5.15 is *undetectable*.

5.4.1 Pole-placement design of full-order observers and compensators

A full-order observer estimates the entire state-vector of a plant, based on the measured output and a known input. If the plant for which the observer is required is linear, the observer's dynamics would also be described by linear state-equations. Consider a noise-free, linear, time-invariant plant described by the following state and output equations:

$$\mathbf{x}^{(1)}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \quad (5.95)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \quad (5.96)$$

The linear, time-invariant state-equation which describes the dynamics of a full-order observer can be expressed as follows:

$$\mathbf{x}_o^{(1)}(t) = \mathbf{A}_o\mathbf{x}_o(t) + \mathbf{B}_o\mathbf{u}(t) + \mathbf{L}\mathbf{y}(t) \tag{5.97}$$

where $\mathbf{x}_o(t)$ is the *estimated state-vector*, $\mathbf{u}(t)$ is the input vector, $\mathbf{y}(t)$ is the output vector, $\mathbf{A}_o$, $\mathbf{B}_o$ are the state-dynamics and control coefficient matrices of the observer, and $\mathbf{L}$ is the *observer gain matrix*. The matrices $\mathbf{A}_o$, $\mathbf{B}_o$, and $\mathbf{L}$ must be selected in a design process such that the *estimation error*, $\mathbf{e}_o(t) = \mathbf{x}(t) - \mathbf{x}_o(t)$, is brought to zero in the steady state. On subtracting Eq. (5.97) from Eq. (5.95), we get the following *error dynamics* state-equation:

$$\mathbf{e}_o^{(1)}(t) = \mathbf{A}_o\mathbf{e}_o(t) + (\mathbf{A} - \mathbf{A}_o)\mathbf{x}(t) + (\mathbf{B} - \mathbf{B}_o)\mathbf{u}(t) - \mathbf{L}\mathbf{y}(t) \tag{5.98}$$

Substitution of Eq. (5.96) into Eq. (5.98) yields

$$\mathbf{e}_o^{(1)}(t) = \mathbf{A}_o\mathbf{e}_o(t) + (\mathbf{A} - \mathbf{A}_o)\mathbf{x}(t) + (\mathbf{B} - \mathbf{B}_o)\mathbf{u}(t) - \mathbf{L}[\mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)] \tag{5.99}$$

or

$$\mathbf{e}_o^{(1)}(t) = \mathbf{A}_o\mathbf{e}_o(t) + (\mathbf{A} - \mathbf{A}_o - \mathbf{L}\mathbf{C})\mathbf{x}(t) + (\mathbf{B} - \mathbf{B}_o - \mathbf{L}\mathbf{D})\mathbf{u}(t) \tag{5.100}$$

From Eq. (5.100), it is clear that estimation error, $\mathbf{e}_o(t)$, will go to zero in the steady state irrespective of $\mathbf{x}(t)$ and $\mathbf{u}(t)$, if all the *eigenvalues* of $\mathbf{A}_o$ are in the *left-half plane*, and the coefficient matrices of $\mathbf{x}(t)$ and $\mathbf{u}(t)$ are *zeros*, i.e. $(\mathbf{A} - \mathbf{A}_o - \mathbf{L}\mathbf{C}) = \mathbf{0}$, $(\mathbf{B} - \mathbf{B}_o - \mathbf{L}\mathbf{D}) = \mathbf{0}$. The latter requirement leads to the following expressions for $\mathbf{A}_o$ and $\mathbf{B}_o$:

$$\mathbf{A}_o = \mathbf{A} - \mathbf{L}\mathbf{C}; \quad \mathbf{B}_o = \mathbf{B} - \mathbf{L}\mathbf{D} \tag{5.101}$$

The error dynamics state-equation is thus the following:

$$\mathbf{e}_o^{(1)}(t) = (\mathbf{A} - \mathbf{L}\mathbf{C})\mathbf{e}_o(t) \tag{5.102}$$

The observer gain matrix, $\mathbf{L}$, must be selected to place all the eigenvalues of $\mathbf{A}_o$ (which are also the poles of the observer) at desired locations in the left-half plane, which implies that the estimation error dynamics given by Eq. (5.102) is *asymptotically stable* (i.e. $\mathbf{e}_o(t) \to \mathbf{0}$ as $t \to \infty$). On substituting Eq. (5.101) into Eq. (5.97), we can write the full-order observer's state-equation as follows:

$$\begin{aligned}\mathbf{x}_o^{(1)}(t) &= (\mathbf{A} - \mathbf{L}\mathbf{C})\mathbf{x}_o(t) + (\mathbf{B} - \mathbf{L}\mathbf{D})\mathbf{u}(t) + \mathbf{L}\mathbf{y}(t) = \mathbf{A}\mathbf{x}_o(t) + \mathbf{B}\mathbf{u}(t)\\ &\quad + \mathbf{L}[\mathbf{y}(t) - \mathbf{C}\mathbf{x}_o(t) - \mathbf{D}\mathbf{u}(t)]\end{aligned} \tag{5.103}$$

Note that Eq. (5.103) approaches Eq. (5.95) in the steady state if $\mathbf{x}_o(t) \to \mathbf{x}(t)$ as $t \to \infty$. Hence, the observer *mirrors* the plant dynamics if the error dynamics is asymptotically stable. The term $[\mathbf{y}(t) - \mathbf{C}\mathbf{x}_o(t) - \mathbf{D}\mathbf{u}(t)]$ in Eq. (5.103) is called the *residual*, and can be expressed as follows:

$$[\mathbf{y}(t) - \mathbf{C}\mathbf{x}_o(t) - \mathbf{D}\mathbf{u}(t)] = \mathbf{C}\mathbf{x}(t) - \mathbf{C}\mathbf{x}_o(t) = \mathbf{C}\mathbf{e}_o(t) \tag{5.104}$$

From Eq. (5.104), it is clear that the residual is also forced to zero in the steady-state if the error dynamics is asymptotically stable.

The observer design process merely consists of selecting $\mathbf{L}$ by pole-placement of the observer. For single-output plants, the pole-placement of the observer is carried out in a manner similar to the pole-placement of *regulators* for *single-input* plants (see Section 5.3.1). For a plant with the characteristic polynomial written as $|s\mathbf{I} - \mathbf{A}| = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$, it can be shown by steps similar to Section 5.3.1 that the observer gain matrix, $\mathbf{L}$, which places the observer's poles such that the observer's characteristic polynomial is $|s\mathbf{I} - \mathbf{A}_o| = s^n + \beta_{n-1}s^{n-1} + \cdots + \beta_1 s + \beta_0$ is given by

$$\mathbf{L} = [(\boldsymbol{\beta} - \mathbf{a})\mathbf{N}'\mathbf{N}^{-1}]^T \tag{5.105}$$

where $\boldsymbol{\beta} = [\,\beta_{n-1};\quad \beta_{n-2};\quad \ldots;\quad \beta_1;\quad \beta_0\,]$, $\mathbf{a} = [\,a_{n-1};\quad a_{n-2};\quad \ldots;\quad a_1;\quad a_0\,]$, $\mathbf{N}$ is the *observability test matrix* of the plant described by Eqs. (5.95) and (5.96), and $\mathbf{N}'$ is the observability test matrix of the plant when it is in the *observer companion form*. Since for single-input, single-output systems, the observer companion form can be obtained from the controller companion form merely by substituting $\mathbf{A}$ by $\mathbf{A}^\mathbf{T}$, $\mathbf{B}$ by $\mathbf{C}^\mathbf{T}$, and $\mathbf{C}$ by $\mathbf{B}^\mathbf{T}$ (see Chapter 3), you can easily show that $\mathbf{N}' = \mathbf{P}'$, where $\mathbf{P}'$ is the *controllability test matrix* of the plant when it is in the *controller companion form*. Thus, we can write

$$\mathbf{L} = [(\boldsymbol{\beta} - \mathbf{a})\mathbf{P}'\mathbf{N}^{-1}]^T \tag{5.106}$$

Recall that $\mathbf{P}'$ is an upper triangular matrix, given by Eq. (5.51).

Example 5.17

Let us try to design a full-order observer for the inverted pendulum on a moving cart (Example 5.9). A state-space representation of the plant is given by Eq. (5.53), with the numerical values of $\mathbf{A}$ and $\mathbf{B}$ given by Eq. (5.54). For this single-input, two-output plant, let us try to design an observer using *only one of the outputs*. If we select the single output to be $y(t) = \theta(t)$, the angular position of the inverted pendulum, the matrices $\mathbf{C}$ and $\mathbf{D}$ are the following:

$$\mathbf{C} = [\,1;\quad 0;\quad 0;\quad 0\,];\qquad \mathbf{D} = 0 \tag{5.107}$$

The first thing to do is to check whether the plant is *observable* with this choice of the output. We do so by the following MATLAB command:

```
>>N = (ctrb(A',C'); rank(N) <enter>

ans =
     2
```

Since the rank of the observability test matrix, $\mathbf{N}$, is 2, i.e. *less than 4*, the order of the plant, the plant is *unobservable* with the angular position of the pendulum as the only output. Hence, we cannot design an observer using $y(t) = \theta(t)$. If we choose $y(t) = x(t)$, the cart's displacement, then the output coefficient matrices are as follows:

$$\mathbf{C} = [\,0;\quad 1;\quad 0;\quad 0\,];\qquad \mathbf{D} = 0 \tag{5.108}$$

On forming the observability test matrix, **N**, with this choice of output, and checking its rank we get

```
>>N = (ctrb(A',C')); rank(N) <enter>

ans =
     4
```

Since now rank $(\mathbf{N}) = 4$, the order of the plant, the plant is observable with $y(t) = x(t)$, and an observer can be designed based on this choice of the output. Let us place the observer poles at $s = -10 \pm 10i$, and $s = -20 \pm 20i$. Then the observer's characteristic polynomial coefficients vector, $\boldsymbol{\beta}$, is calculated as follows:

```
>>v = [-10-10i -10+10i -20-20i -20+20i]'; p = poly(v); beta = p(2:5)
  <enter>

beta =
     60   1800   24000   160000
```

The plant's characteristic polynomial coefficient vector, **a**, is calculated as follows:

```
>>p = poly(A); a = p(2:5) <enter>

a =
                    0 -10.7800 0 0
```

and the matrix $\mathbf{P}'$ is evaluated using Eq. (5.51) as follows:

```
>>Pdash = [1 -a(1:3); 0 1 -a(1:2); 0 0 1 -a(1); 0 0 0 1] <enter>

Pdash =
         1.0000   0        10.7800    0
         0        1.000    0          10.7800
         0        0        1.0000     0
         0        0        0          1.0000
```

Finally, the observer gain matrix, **L**, is calculated using Eq. (5.106) as follows:

```
>>format long e; L = ((beta-a)*Pdash*inv(N))' <enter>

L =
 -2.514795918367350e+004
  6.000000000000000e+001
 -1.831838861224490e+005
  1.810780000000000e+003
```

Note that we have printed out **L** in the *long format*, since we need to store it for later calculations. Let us check whether the observer poles have been placed at desired locations, by calculating the eigenvalues of $\mathbf{A}_o = (\mathbf{A} - \mathbf{LC})$ as follows:

```
>>Ao = A-L*C; eig(Ao) <enter>

ans =
 -20.0000+20.0000i
 -20.0000-20.0000i
 -10.0000+10.0000i
 -10.0000-10.0000i
```

Hence, observer pole-placement has been accurately achieved.

Example 5.17 illustrates the ease by which single-output observers can be designed. However, it is impossible to design single-output observers for those plants which are *unobservable* with *any single* output. When *multi-output* observers are required, generally there are more design parameters (i.e. elements in the observer gain matrix, **L**) than the observer poles, hence all of these parameters cannot be determined by pole-placement alone. As in the design of regulators for multi-input plants (Section 5.3.2), additional conditions are required to be satisfied by multi-output observers, apart from pole-placement, to determine the observer gain matrix. These additional conditions are hard to come by, and thus pole-placement is not a good method of designing multi-output observers. A better design procedure in such cases is the *Kalman filter* approach of Chapter 7.

MATLAB's Control System Toolbox (CST) provides the command *estim* for constructing a state-space model, *syso*, of the observer with the observer gain matrix, **L**, and a state-space model, *sysp*, of the plant, with state coefficient matrices **A, B, C, D**, as follows:

```
>>sysp=ss[A,B,C,D]; sysp = estim(syso,L) <enter>
```

The input to the observer thus formed is the plant's output, $\mathbf{y}(t)$, while output vector of the observer is $[\{\mathbf{C}\mathbf{x}_o(t)\}^T; \quad \mathbf{x}_o(t)^T]^T$, where $\mathbf{x}_o(t)$ is the estimated state-vector.

Observers (also known as *estimators*) by themselves are very useful in estimating the plant dynamics from a limited number of outputs, and are employed in *parameter estimation, fault detection*, and other similar applications. The utility of an observer in a control system lies in feeding the estimated state-vector to a controller for generating input signals for the plant. The controllers which generate input signals for the plant based on the estimated state-vector (rather than the actual, fed back state-vector) are called *compensators*. However, design of compensators involves a dilemma. The estimated state-vector is obtained from an observer, which treats the plant's input vector as a *known quantity*, while the compensator is *yet to* generate the input vector based on the estimated state-vector. It is like the classic chicken and egg problem, since we do not know which came first: the control input on which the estimated state-vector is based, or the estimated state-vector on which the input is based! A practical way of breaking this vicious circle is the *separation principle*, which states that if we design an observer (assuming known input vector), and a compensator (assuming known estimated state-vector) *separately*, and then combine the two, we will end up with a control system *that works*. The separation principle thus allows us to design the observer and the controller *independently of each* *other*. The resulting control system can be a *regulator* or a *tracking system*, depending on the desired state-vector being *zero* or *non-zero*, respectively.

Let us consider a tracking system (i.e. a control system with a non-zero desired state-vector) based on a noise-free plant described by Eqs. (5.95) and (5.96), for which a full-order observer, given by Eq. (5.103) has been designed. Then a compensator can be designed to generate the input vector for the plant according to the following control-law:

$$\mathbf{u}(t) = \mathbf{K}[\mathbf{x}_d(t) - \mathbf{x}_o(t)] - \mathbf{K}_d\mathbf{x}_d(t) \tag{5.109}$$

where $\mathbf{x}_o(t)$ is the *estimated state-vector*, $\mathbf{x}_d(t)$ is the *desired state-vector*, $\mathbf{K}$ is the *feedback gain matrix*, and $\mathbf{K}_d$ is the *feedforward gain matrix*. On substituting Eq. (5.109) into Eq. (5.103), the observer state-equation becomes

$$\mathbf{x}_o^{(1)}(t) = (\mathbf{A} - \mathbf{LC} - \mathbf{BK} + \mathbf{LDK})\mathbf{x}_o(t) + (\mathbf{B} - \mathbf{LD})(\mathbf{K} - \mathbf{K}_d)\mathbf{x}_d(t) + \mathbf{Ly}(t) \tag{5.110}$$

On substituting the output equation, Eq. (5.96), into Eq. (5.110), and again substituting Eq. (5.109), we get the following state-equation for the compensator:

$$\mathbf{x}_o^{(1)}(t) = (\mathbf{A} - \mathbf{LC} - \mathbf{BK})\mathbf{x}_o(t) + \mathbf{B}(\mathbf{K} - \mathbf{K}_d)\mathbf{x}_d(t) + \mathbf{LCx}(t) \tag{5.111}$$

The plant's state-equation, Eq. (5.95), when the input is given by Eq. (5.109), becomes the following:

$$\mathbf{x}^{(1)}(t) = \mathbf{Ax}(t) - \mathbf{BKx}_o(t) + \mathbf{B}(\mathbf{K} - \mathbf{K}_d)\mathbf{x}_d(t) \tag{5.112}$$

Equations. (5.111) and (5.112) are the state-equations of the closed-loop system, and can be expressed as follows:

$$\begin{bmatrix} \mathbf{x}^{(1)}(t) \\ \mathbf{x}_o^{(1)}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & -\mathbf{BK} \\ \mathbf{LC} & (\mathbf{A} - \mathbf{LC} - \mathbf{BK}) \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{x}_o(t) \end{bmatrix} + \begin{bmatrix} \mathbf{B}(\mathbf{K} - \mathbf{K}_d) \\ \mathbf{B}(\mathbf{K} - \mathbf{K}_d) \end{bmatrix} \mathbf{x}_d(t) \tag{5.113}$$

The closed-loop tracking system is thus of order $2n$, where n is the order of the plant. The input to the closed-loop system is the desired state-vector, $\mathbf{x}_d(t)$. A schematic diagram of the tracking system is shown in Figure 5.16. Note that this control system is essentially

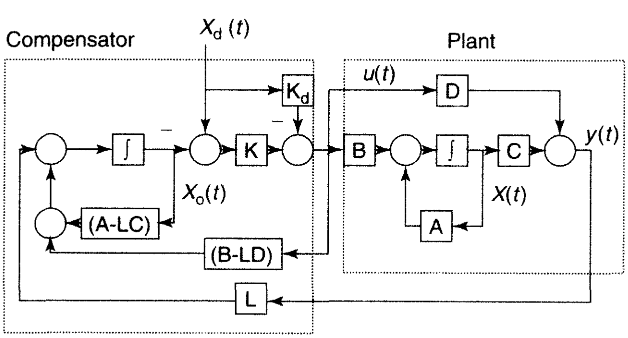

Figure 5.16 Closed-loop tracking system with a full-order compensator

based on the feedback of the output vector, $\mathbf{y}(t)$, to the compensator, which generates the input vector, $\mathbf{u}(t)$, for the plant.

To obtain the state-equation for the estimation error, $\mathbf{e}_o(t) = \mathbf{x}(t) - \mathbf{x}_o(t)$, let us write Eq. (5.112) as follows:

$$\mathbf{x}^{(1)}(t) = (\mathbf{A} - \mathbf{BK})\mathbf{x}(t) + \mathbf{BKe}_o(t) + \mathbf{B}(\mathbf{K} - \mathbf{K}_d)\mathbf{x}_d(t) \tag{5.114}$$

On subtracting Eq. (5.111) from Eq. (5.114) we get

$$\mathbf{e}_o^{(1)}(t) = (\mathbf{A} - \mathbf{LC})\mathbf{e}_o(t) \tag{5.115}$$

which is the same as Eq. (5.102). The state-equation for the tracking error, $\mathbf{e}(t) = \mathbf{x}_d(t) - \mathbf{x}(t)$, is obtained by subtracting Eq. (5.114) from Eq. (5.74), which results in

$$\mathbf{e}^{(1)}(t) = (\mathbf{A} - \mathbf{BK})\mathbf{e}(t) + (\mathbf{A}_d - \mathbf{A} + \mathbf{BK}_d)\mathbf{x}_d(t) - \mathbf{BKe}_o(t) \tag{5.116}$$

The tracking system's error dynamics is thus represented by Eqs. (5.115) and (5.116), which can be expressed together as follows:

$$\begin{bmatrix} \mathbf{e}^{(1)}(t) \\ \mathbf{e}_o^{(1)}(t) \end{bmatrix} = \begin{bmatrix} (\mathbf{A} - \mathbf{BK}) & -\mathbf{BK} \\ \mathbf{0} & (\mathbf{A} - \mathbf{LC}) \end{bmatrix} \begin{bmatrix} \mathbf{e}(t) \\ \mathbf{e}_o(t) \end{bmatrix} + \begin{bmatrix} (\mathbf{A}_d - \mathbf{A} + \mathbf{BK}_d) \\ \mathbf{0} \end{bmatrix} \mathbf{x}_d(t) \tag{5.117}$$

Note that Eq. (5.117) represents the closed-loop tracking system in a *decoupled* state-space form. The closed-loop poles must be the eigenvalues of the following closed-loop state-dynamics matrix, $\mathbf{A}_{\mathbf{CL}}$:

$$\mathbf{A}_{\mathbf{CL}} = \begin{bmatrix} (\mathbf{A} - \mathbf{BK}) & \mathbf{0} \\ \mathbf{0} & (\mathbf{A} - \mathbf{LC}) \end{bmatrix} \tag{5.118}$$

Equation (5.117) implies that the closed-loop poles are the eigenvalues of $\mathbf{A}_{\mathbf{CL}}$, i.e. the roots of the characteristic equation $|s\mathbf{I} - \mathbf{A}_{\mathbf{CL}}| = 0$, which can be written as $|[s\mathbf{I} - (\mathbf{A} - \mathbf{BK})][s\mathbf{I} - (\mathbf{A} - \mathbf{LC})]| = 0$, resulting in $|s\mathbf{I} - (\mathbf{A} - \mathbf{BK})| = 0$ and $|s\mathbf{I} - (\mathbf{A} - \mathbf{LC})| = 0$. Hence, the *closed-loop poles* are the *eigenvalues of* $(\mathbf{A} - \mathbf{BK})$ and *eigenvalues of* $(\mathbf{A} - \mathbf{LC})$, which are also the *poles of* the full-state feedback *regulator* and the *observer*, respectively. Note from Eq. (5.117) that for the estimation error, $\mathbf{e}_o(t)$, to go to zero in the steady state, all the eigenvalues of $(\mathbf{A} - \mathbf{LC})$ must be in the left-half plane. Also, for the tracking error, $\mathbf{e}(t)$, to go to zero in the steady state, irrespective of the desired state-vector, $\mathbf{x}_d(t)$, all the eigenvalues of $(\mathbf{A} - \mathbf{BK})$ must be in the left-half plane, and the coefficient matrix multiplying $\mathbf{x}_d(t)$ must be zero, $(\mathbf{A}_d - \mathbf{A} + \mathbf{BK}_d) = \mathbf{0}$. Recall from Section 5.3 that $(\mathbf{A} - \mathbf{BK})$ is the state-dynamics matrix of the *full-state feedback regulator*, and from Eq. (5.103) that $(\mathbf{A} - \mathbf{LC})$ is the state-dynamics matrix of the *full-order observer*. Hence, the compensator design process consists of *separately* deriving the feedback gain matrices $\mathbf{L}$ and $\mathbf{K}$, by pole-placement of the observer and the full-state feedback regulator, respectively, and selecting $\mathbf{K}_d$ to satisfy $(\mathbf{A}_d - \mathbf{A} + \mathbf{BK}_d) = \mathbf{0}$. Usually, it is impossible to satisfy $(\mathbf{A}_d - \mathbf{A} + \mathbf{BK}_d) = \mathbf{0}$ by selecting the feedforward gain matrix, $\mathbf{K}_d$. Alternatively, it may be possible to satisfy $(\mathbf{A}_d - \mathbf{A} + \mathbf{BK}_d)\mathbf{x}_d(t) = \mathbf{0}$ when some elements of $\mathbf{x}_d(t)$ are zeros. Hence, the steady state tracking error can generally be reduced to zero only for some values of the desired state-vector. In the above steps, we have assumed that the desired state-vector, $\mathbf{x}_d(t)$, is available for measurement. In many cases, it is possible to measure only a desired output, $\mathbf{y}_d(t) = \mathbf{C}_d\mathbf{x}_d(t)$, rather than $\mathbf{x}_d(t)$ itself. In such cases, an observer can be designed to estimate $\mathbf{x}_d(t)$ based on the measurement of the desired output. It is left to you as an exercise to derive the state-equations for the compensator when $\mathbf{x}_d(t)$ is not measurable.

Example 5.18

Let us design a compensator for the inverted pendulum on a moving cart (Example 5.9), when it is desired to move the cart by 1 m, while not letting the pendulum fall. Such a tracking system is representative of a *robot*, which is bringing to you an inverted champagne bottle precariously balanced on a finger! The plant is clearly unstable (as seen in Example 5.9). The task of the compensator is to stabilize the inverted pendulum, while moving the cart by the desired displacement. The desired state-vector is thus a constant, consisting of the desired angular position of the inverted pendulum, $\theta_d(t) = 0$, desired cart displacement, $x_d(t) = 1$ m, desired angular velocity of the pendulum, $\theta_d^{(1)}(t) = 0$, and desired cart velocity, $x_d^{(1)}(t) = 0$. Hence, $\mathbf{x}_d(t) = [0; 1; 0; 0]^T$. Since $\mathbf{x}_d(t)$ is constant, it implies that $\mathbf{x}_d^{(1)}(t) = \mathbf{0}$, and from Eq. (5.74), $\mathbf{A}_d = \mathbf{0}$. By the separation principle, we can design a tracking system *assuming full-state feedback*, and then combine it with a full-order observer, which estimates the plant's state-vector. A full-state feedback regulator has already been designed for this plant in Example 5.11, which places the eigenvalues of the regulator state-dynamics matrix, $(\mathbf{A} - \mathbf{BK})$, at $s = -7.853 \pm 3.2528i$, and $s = -7.853 \pm 7.853i$ using the following feedback gain matrix:

$$\mathbf{K} = [\,-1362.364050360232;\ \ -909.3160795202226;\ \ -344.8741667548096;\ \ -313.4621667548089\,] \tag{5.119}$$

We have also designed a full-order observer for this plant using the cart displacement, $x(t)$, as the output in Example 5.17. The observer poles, i.e. the eigenvalues of $(\mathbf{A} - \mathbf{LC})$, were selected to be at $s = -10 \pm 10i$, and $s = -20 \pm 20i$, and the observer gain matrix which achieved this observer pole configuration was obtained to be the following:

$$\mathbf{L} = [\,-25\,149.79591836735;\quad 60.0;\quad -183\,183.8861224490;\quad 1810.780\,]^T \tag{5.120}$$

The separation principle allows us to combine the separately designed observer and regulator into a compensator. However, it remains for us to determine the feedforward gain matrix, $\mathbf{K}_d$. The design requirement of zero tracking error in the steady state is satisfied if $(\mathbf{A}_d - \mathbf{A} + \mathbf{BK}_d)\mathbf{x}_d(t) = \mathbf{0}$ in Eq. (5.117). The elements of

$\mathbf{K}_d = [\,K_{d1};\quad K_{d2};\quad K_{d3};\quad K_{d4}\,]$ are thus determined as follows:

$$(\mathbf{A}_d - \mathbf{A} + \mathbf{B}\mathbf{K}_d)\mathbf{x}_d(t) = \begin{bmatrix} 0 \\ 0 \\ K_{d2} \\ K_{d2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{5.121}$$

Equation (5.121) is exactly satisfied by selecting $K_{d2} = 0$. What about the other elements of $\mathbf{K}_d$? There are no conditions placed on the other elements of $\mathbf{K}_d$, and thus we can *arbitrarily* take them to be zeros. Therefore, by choosing $\mathbf{K}_d = \mathbf{0}$, we are able to meet the zero tracking error requirement in the steady state. On substituting the designed values of the gain matrices, $\mathbf{K}$, $\mathbf{L}$, and $\mathbf{K}_d$ into Eq. (5.113), we can get the closed-loop state-equations for the tracking system in terms of the plant's state-vector, $\mathbf{x}(t)$, and the estimated state-vector, $\mathbf{x}_o(t)$, and then solve them to get the closed-loop response. This is done using MATLAB as follows:

```
>>K=[-1362.364050360232 -909.3160795202226 -344.8741667548096 -313.46216675
  48089]; <enter>

>>L=[-25149.79591836735 60.0 -183183.8861224490 1810.780]'; Kd=zeros(1,4);
  <enter>

>>ACL = [A -B*K; L*C (A-L*C-B*K)]; BCL = [B*(K-Kd); B*(K-Kd)];<enter>
```

Let us confirm that the eigenvalues of $\mathbf{A}_{CL}$ are the poles of the regulator designed in Example 5.11 and the observer designed in Example 5.17 as follows:

```
>>eig(ACL) <enter>

ans =
-20.0000+20.0000i
-20.0000-20.0000i
-10.0000+10.0000i
-10.0000-10.0000i
-7.8530+7.8530i
-7.8530-7.8530i
-7.8530+3.2528i
-7.8530-3.2528i
```

which indeed they are. Finally, the closed-loop response to the desired state-vector is calculated as follows:

```
>>t = 0:1.0753e-2:1.2; n=size(t,2); for i=1:n; Xd(i,:) = [0 1 0 0]; end
  <enter>

>>sysCL=ss(ACL, BCL,[C zeros(1,4)],zeros(1,4)); [y,t,X] = lsim(sysCL,Xd,t');
  <enter>
```

The closed-loop cart's displacement, $x(t)$, and pendulum's angular position, $\theta(t)$, are plotted in Figure 5.17, as follows:

```
>>plot(t,X(:,1:2)) <enter>
```

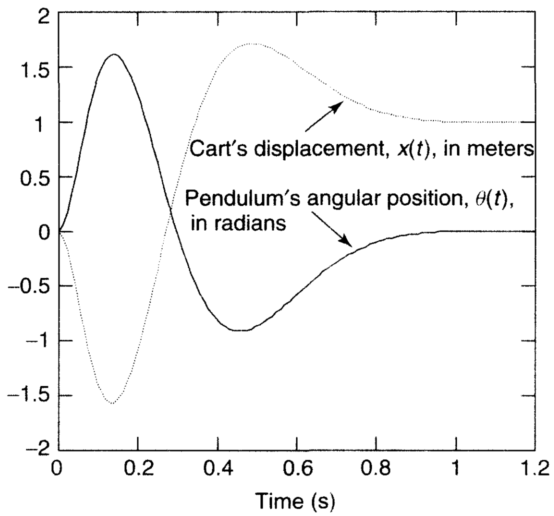

Figure 5.17 Response of the compensator based tracking system for inverted- pendulum on a moving cart, with desired angular position, $\theta_d(t) = 0$, and desired cart's displacement, $x_d(t) = 1$ m, when the regulator poles are $s = -7.853 \pm 3.2528i$, and $s = -7.853 \pm 7.853i$

The closed-loop transient response for $x(t)$ and $\theta(t)$ is seen in Figure 5.17 to settle to their respective desired values in about 1 s, with maximum overshoots of 1.65 m and 1.57 rad., respectively. However, an overshoot of 1.57 rad. corresponds to 90°, which implies that the pendulum *has been allowed to fall* and then brought back up to the inverted position, $\theta(t) = 0°$. If the inverted pendulum represents a drink being brought to you by a robot (approximated by the moving cart), clearly this compensator design would be unacceptable, and it will be necessary to reduce the maximum overshoot to an angle less than 90° by suitably modifying the closed-loop poles. Recall from Example 3.3 that the linearized state-space model of the system given by Eq. (5.53) is *invalid* when the pendulum sways by a large angle, $\theta(t)$, and the results plotted in Figure 5.17 are thus *inaccurate*. Hence, the regulator design that was adequate for stabilizing the plant in the presence of a *small* initial disturbance in cart displacement, is *unsatisfactory* for moving the cart by a *large* displacement. Note that the location of the regulator poles, i.e. the eigenvalues of $(\mathbf{A} - \mathbf{BK})$, governs the closed-loop response of the plant's state-vector, $\mathbf{x}(t)$. By moving the regulator poles closer to the imaginary axis, it would be possible to reduce the maximum overshoot *at the cost of increased settling time*. Let us select the new regulator poles as $s = -0.7853 \pm 3.25328i$ and $s = -0.7853 \pm 0.7853i$. The new feedback gain matrix, $\mathbf{K}$, is calculated as follows:

```
>>v=[-0.7853+3.25328i -0.7853-3.25328i -0.7853+0.7853i -0.7853-0.7853i]';
  K=place(A,B,v)
place: ndigits= 16

K =
  -27.0904    -1.4097    -5.1339    -1.9927
```

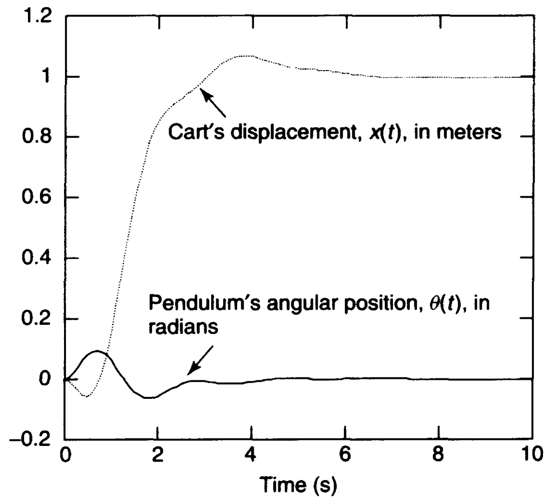

Figure 5.18 Response of the compensator based tracking system for inverted- pendulum on a moving cart, with desired angular position, $\theta_d(t) = 0$, and desired cart's displacement, $x_d(t) = 1$ m, when regulator poles are $s = -0.7853 \pm 3.25328i$ and $s = -0.7853 \pm 0.7853i$

and the new closed-loop response is plotted in Figure 5.18, which shows that the maximum overshoots have been reduced to less than 1.1 m and 0.1 rad. (5.7°) for $x(t)$ and $\theta(t)$, respectively, but the settling time is increased to about 7 s. Since the pendulum now sways by small angles, the linearized model of Eq. (5.53) is valid, and the compensator design is acceptable. However, the robot now takes 7 seconds in bringing your drink placed 1 m away! You may further refine the design by experimenting with the regulator pole locations.

Let us see how well the compensator estimates the state-vector by looking at the estimation error vector, $\mathbf{e}_o(t) = \mathbf{x}(t) - \mathbf{x}_o(t)$. The elements of the estimation error vector, $e_{o1}(t) = \theta_d(t) - \theta(t)$, $e_{o2}(t) = x_d(t) - x(t)$, $e_{o3}(t) = \theta_d^{(1)}(t) - \theta^{(1)}(t)$, and $e_{o4}(t) = x_d^{(1)}(t) - x^{(1)}(t)$ are plotted in Figure 5.19 as follows:

```
>>plot(t,X(:,1)-X(:,5),t,X(:,2)-X(:,6),t,X(:,3)-X(:,7),t,X(:,4)-X(:,8))
  <enter>
```

Figure 5.19 shows that the largest estimation error magnitude is about 1.5×10^{-9} rad/s for estimating the pendulum's angular velocity, $\theta^{(1)}(t)$, and about 5×10^{-10} rad. for estimating the pendulum's angular position, $\theta(t)$. Since the observer is based on the measurement of the cart's displacement, $x(t)$, the estimation error magnitudes of $x(t)$ and $x^{(1)}(t)$ are seen to be negligible in comparison with those of $\theta(t)$ and $\theta^{(1)}(t)$. All the estimation errors decay to zero in about 7 s, which is the same time as the settling time of the closed-loop response for the state-vector, $\mathbf{x}(t)$. The observer poles are therefore at acceptable locations. Note that we can move the observer poles as much inside the left-half plane as we want, because there is no control input cost associated with the observer. However, if the measurements of the output are noisy, there will be an increased influence

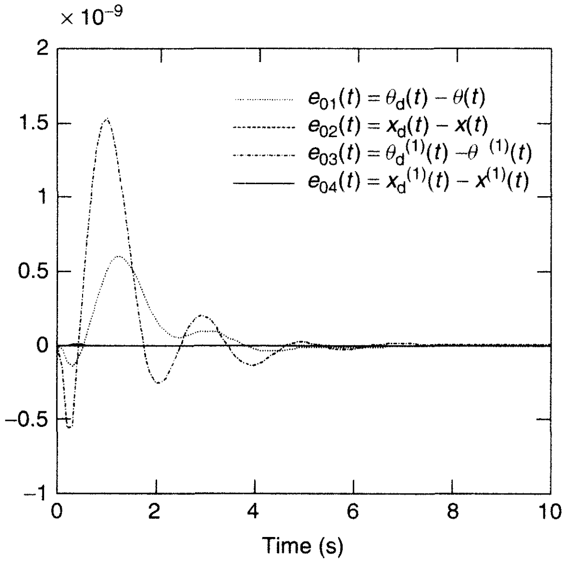

Figure 5.19 Estimation errors for the compensator based tracking system for inverted- pendulum on a moving cart, with desired angular position, $\theta_d(t) = 0$, and desired cart's displacement, $x_d(t) = 1$ m, when regulator poles are $s = -0.7853 \pm 3.25328i$ and $s = -0.7853 \pm 0.7853i$

of noise on the closed-loop system if the observer poles are too far inside the left-half plane.

5.4.2 Pole-placement design of reduced-order observers and compensators

When some of the state variables of a plant can be measured, it is unnecessary to estimate those state variables. Hence, a *reduced-order observer* can be designed which estimates only those state variables that cannot be measured. Suppose the state-vector of a plant, $\mathbf{x}(t)$, can be partitioned into a vector containing measured state variables, $\mathbf{x}_1(t)$, and unmeasurable state variables, $\mathbf{x}_2(t)$, i.e. $\mathbf{x}(t) = [\mathbf{x}_1(t)^T; \ \mathbf{x}_2(t)^T]^T$. The measured output vector, $\mathbf{y}(t)$, may either be equal to the vector, $\mathbf{x}_1(t)$ – implying that all the state variables constituting $\mathbf{x}_1(t)$ can be *directly* measured – or it may be equal to a *linear combination* of the state variables constituting $\mathbf{x}_1(t)$. Hence, the output equation can be generally expressed as

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}_1(t) \tag{5.122}$$

where $\mathbf{C}$ is a constant, *square* matrix, indicating that there are as many outputs as the number of elements in $\mathbf{x}_1(t)$. When $\mathbf{x}_1(t)$ can be directly measured, $\mathbf{C} = \mathbf{I}$. The plant's state-equation (Eq. (5.95)) can be expressed in terms of the partitioned state-vector,

$\mathbf{x}(t) = [\,\mathbf{x}_1(t)^T; \quad \mathbf{x}_2(t)^T\,]^T$, as follows:

$$\mathbf{x}_1^{(1)}(t) = \mathbf{A}_{11}\mathbf{x}_1(t) + \mathbf{A}_{12}\mathbf{x}_2(t) + \mathbf{B}_1\mathbf{u}(t) \tag{5.123}$$

$$\mathbf{x}_2^{(1)}(t) = \mathbf{A}_{12}\mathbf{x}_1(t) + \mathbf{A}_{22}\mathbf{x}_2(t) + \mathbf{B}_2\mathbf{u}(t) \tag{5.124}$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix} \tag{5.125}$$

Let the order of the plant be n, and the number of measured state variables (i.e. the dimension of $\mathbf{x}_1(t)$) be k. Then a reduced-order observer is required to estimate the vector $\mathbf{x}_2(t)$, which is of dimension $(n-k)$. Hence, the estimated state-vector is simply given by

$$\mathbf{x}_o(t) = \begin{bmatrix} \mathbf{x}_1(t) \\ \mathbf{x}_{o2}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{C}^{-1}\mathbf{y}(t) \\ \mathbf{x}_{o2}(t) \end{bmatrix} \tag{5.126}$$

where $\mathbf{x}_{o2}(t)$ is the *estimation* of the vector $\mathbf{x}_2(t)$. Note that Eq. (5.126) requires that $\mathbf{C}$ should be a *non-singular* matrix, which implies that the plant should be *observable* with the output given by Eq. (5.122). If the plant is *unobservable* with the output given by Eq. (5.122), $\mathbf{C}$ would be *singular*, and a reduced-order observer *cannot* be designed.

The observer state-equation should be such that the *estimation error*, $\mathbf{e}_{o2}(t) = \mathbf{x}_2(t) - \mathbf{x}_{o2}(t)$, is always brought to zero in the steady state. A possible observer state-equation would appear to be the extension of the full-order observer state-equation (Eq. (5.103)) for the reduced-order observer, written as follows:

$$\mathbf{x}_o^{(1)}(t) = \mathbf{A}\mathbf{x}_o(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{L}[\mathbf{y}(t) - \mathbf{C}\mathbf{x}_o(t)] \tag{5.127}$$

where the observer gain matrix, $\mathbf{L}$, would determine the estimation error dynamics. On substituting Eq. (5.126) into Eq. (5.127), and subtracting the resulting state-equation for $\mathbf{x}_{o2}(t)$ from Eq. (5.124), we can write the estimation error state-equation as follows:

$$\mathbf{e}_{o2}^{(1)}(t) = \mathbf{A}_{22}\mathbf{e}_{o2}(t) \tag{5.128}$$

However, Eq. (5.128) indicates that the estimation error is *unaffected* by the observer gain matrix, $\mathbf{L}$, and solely depends upon the plant's sub-matrix, $\mathbf{A}_{22}$. If $\mathbf{A}_{22}$ turns out to be a matrix having eigenvalues with *positive* real parts, we will be stuck with an estimation error that goes to *infinity* in the steady state! Clearly, the observer state-equation given by Eq. (5.127) is *unacceptable*. Let us try the following reduced-order observer dynamics:

$$\mathbf{x}_{o2}(t) = \mathbf{L}\mathbf{y}(t) + \mathbf{z}(t) \tag{5.129}$$

where $\mathbf{z}(t)$ is the solution of the following state-equation:

$$\mathbf{z}^{(1)}(t) = \mathbf{F}\mathbf{z}(t) + \mathbf{H}\mathbf{u}(t) + \mathbf{G}\mathbf{y}(t) \tag{5.130}$$

Note that the reduced-order observer gain matrix, $\mathbf{L}$, defined by Eq. (5.129), is of size $[(n-k) \times k]$, whereas the full-order observer gain matrix would be of size $(n \times k)$. On differentiating Eq. (5.129) with respect to time, subtracting the result from Eq. (5.124), and substituting $\mathbf{z}(t) = \mathbf{x}_{o2}(t) - \mathbf{L}\mathbf{y}(t) = \mathbf{x}_2(t) - \mathbf{e}_{o2}(t) - \mathbf{L}\mathbf{C}\mathbf{x}_1(t)$, the state-equation for estimation error is written as follows:

$$\begin{aligned}\mathbf{e}_{o2}^{(1)}(t) = {} & \mathbf{F}\mathbf{e}_{o2}(t) + (\mathbf{A}_{21} - \mathbf{LCA}_{11} + \mathbf{FLC})\mathbf{x}_1(t) + (\mathbf{A}_{22} - \mathbf{LCA}_{12} - \mathbf{F})\mathbf{x}_2(t) \\ & + (\mathbf{B}_2 - \mathbf{LCB}_1 - \mathbf{H})\mathbf{u}(t)\end{aligned} \tag{5.131}$$

Equation (5.131) implies that for the estimation error, $\mathbf{e}_{o2}(t)$, to go to zero in the steady state, irrespective of $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, and $\mathbf{u}(t)$, the coefficient matrices multiplying $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, and $\mathbf{u}(t)$ *must vanish*, and $\mathbf{F}$ must have *all* eigenvalues in the left-half plane. Therefore, it follows that

$$\mathbf{F} = \mathbf{A}_{22} - \mathbf{LCA}_{12}; \quad \mathbf{H} = \mathbf{B}_2 - \mathbf{LCB}_1; \quad \mathbf{G} = \mathbf{FL} + (\mathbf{A}_{21} - \mathbf{LCA}_{11})\mathbf{C}^{-1} \tag{5.132}$$

The reduced-order observer design consists of selecting the observer gain matrix, $\mathbf{L}$, such that all the eigenvalues of $\mathbf{F}$ are in the left-half plane.

Example 5.19

Let us design a reduced-order observer for the inverted pendulum on a moving cart (Example 5.9), based on the measurement of the cart displacement, $x(t)$. The first step is to partition the state-vector into measurable and unmeasurable parts, i.e. $\mathbf{x}(t) = [\mathbf{x}_1(t)^T; \quad \mathbf{x}_2(t)^T]^T$, where $\mathbf{x}_1(t) = x(t)$, and $\mathbf{x}_2(t) = [\theta(t); \quad \theta^{(1)}(t); \quad x^{(1)}(t)]^T$. However, in Example 5.9, the state-vector was expressed as $[\theta(t); \quad x(t); \quad \theta^{(1)}(t); \quad x^{(1)}(t)]^T$. We must therefore rearrange the state coefficient matrices (Eq. (5.54)) such that the state-vector is $\mathbf{x}(t) = [x(t); \quad \theta(t); \quad \theta^{(1)}(t); \quad x^{(1)}(t)]^T$ and partition them as follows:

$$\mathbf{A} = \left[\begin{array}{c:ccc} 0 & 0 & 0 & 1 \\ \hdashline 0 & 0 & 1 & 0 \\ 0 & 10.78 & 0 & 0 \\ 0 & -0.98 & 0 & 0 \end{array}\right]; \qquad \mathbf{B} = \left[\begin{array}{c} 0 \\ \hdashline 0 \\ -1 \\ 1 \end{array}\right] \tag{5.133}$$

From Eq. (5.133) it is clear that

$$\mathbf{A}_{11} = 0; \quad \mathbf{A}_{12} = [0 \quad 0 \quad 1]; \quad \mathbf{B}_1 = 0$$

$$\mathbf{A}_{21} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}; \quad \mathbf{A}_{22} = \begin{bmatrix} 0 & 1 & 0 \\ 10.78 & 0 & 0 \\ -0.98 & 0 & 0 \end{bmatrix}; \quad \mathbf{B}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \tag{5.134}$$

Since the measured output is $\mathbf{x}_1(t) = x(t)$, the output equation is $\mathbf{y}(t) = \mathbf{C}\mathbf{x}_1(t)$, where $\mathbf{C} = 1$. We have to select an observer gain-matrix, $\mathbf{L}$, such that the eigenvalues of $\mathbf{F} = (\mathbf{A}_{22} - \mathbf{L}\mathbf{C}\mathbf{A}_{12})$ are in the left-half plane. Let us select the observer poles, i.e. the eigenvalues of $\mathbf{F}$, to be $s = -20$, $s = -20 \pm 20i$. Then $\mathbf{L}$ is calculated by pole-placement as follows:

```
>>A12 = [0 0 1]; A22 = [0 1 0; 10.78 0 0; -0.98 0 0]; C = 1; <enter>

>>v = [-20 -20+20i -20-20i]'; L = (place(A22',A12'*C',v))' <enter>

L =
-1.6437e+003
-1.6987e+004
6.0000e+001
```

Therefore, the observer dynamics matrix, $\mathbf{F}$, is calculated as follows:

```
>>F = A22 - L*C*A12 <enter>

F =
  0              1.0000e+000   1.6437e+003
  1.0780e+001    0             1.6987e+004
 -9.8000e-001    0             -6.0000e+001
```

Let us verify that the eigenvalues of $\mathbf{F}$ are at desired locations:

```
>>eig(F) <enter>

ans =
-2.0000e+001+2.0000e+001i
-2.0000e+001-2.0000e+001i
-2.0000e+001
```

which indeed they are. The other observer coefficient matrices, $\mathbf{G}$ and $\mathbf{H}$, are calculated as follows:

```
>>A11 = 0; A21 = [0 0 0]'; B1 = 0; B2 = [0 -1 1]';
   H = B2 - L*C*B1 <enter>

H =
        0
        -1
        1

>>G = F*L + (A21 - L*C*A11)*inv(C) <enter>

G =
        8.1633e+004
        1.0015e+006
        -1.9892e+003
```

A compensator based on the reduced-order observer can be designed by the *separation principle*, in a manner similar to the compensator based on the full-order observer. The control-law defining the reduced-order compensator for a tracking system can be expressed as follows, after substituting Eq. (5.126) into Eq. (5.109):

$$\mathbf{u}(t) = \mathbf{K}[\mathbf{x}_d(t) - \mathbf{x}_o(t)] - \mathbf{K}_d\mathbf{x}_d(t) = (\mathbf{K} - \mathbf{K}_d)\mathbf{x}_d(t) - \mathbf{K}_1\mathbf{x}_1(t) - \mathbf{K}_2\mathbf{x}_{o2}(t) \quad (5.135)$$

where $\mathbf{x}_d(t)$ is the desired state-vector, $\mathbf{K}_d$ is the feedforward gain matrix, and $\mathbf{K}$ is the feedback gain matrix, which can be partitioned into gain matrices that feedback $\mathbf{x}_1(t)$ and $\mathbf{x}_{o2}(t)$, respectively, as $\mathbf{K} = [\,\mathbf{K}_1; \quad \mathbf{K}_2\,]$. A schematic diagram of the reduced-order compensator is shown in Figure 5.20.

The estimation error dynamics of the reduced-order compensator is described by the following state-equation, obtained by substituting Eq. (5.132) into Eq. (5.131):

$$\mathbf{e}_{o2}{}^{(1)}(t) = \mathbf{F}\mathbf{e}_{o2}(t) \quad (5.136)$$

while the state-equation for the tracking error, $\mathbf{e}(t) = \mathbf{x}_d(t) - \mathbf{x}(t)$, is obtained by subtracting Eq. (5.74) from Eq. (5.95), and substituting Eq. (5.135) as follows:

$$\mathbf{e}^{(1)}(t) = \mathbf{A}\mathbf{e}(t) + (\mathbf{A}_d - \mathbf{A} + \mathbf{B}\mathbf{K}_d)\mathbf{x}_d(t) - \mathbf{B}\mathbf{K}[\mathbf{x}_d(t) - \mathbf{x}_o(t)] \quad (5.137)$$

On substituting for $\mathbf{x}_o(t)$ from Eq. (5.126), Eq. (5.137) can be written as follows:

$$\mathbf{e}^{(1)}(t) = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{e}(t) + (\mathbf{A}_d - \mathbf{A} + \mathbf{B}\mathbf{K}_d)\mathbf{x}_d(t) - \mathbf{B}\mathbf{K}_2\mathbf{e}_{o2}(t) \quad (5.138)$$

Hence, the dynamics of the tracking system can be described by Eqs. (5.136) and (5.138). To have the tracking error go to zero in the steady state, irrespective of $\mathbf{x}_d(t)$, we must select the feedforward gain matrix, $\mathbf{K}_d$, such that $(\mathbf{A}_d - \mathbf{A} + \mathbf{B}\mathbf{K}_d)\mathbf{x}_d(t) = \mathbf{0}$, and the feedback gain matrix, $\mathbf{K}$, such that the eigenvalues of $(\mathbf{A} - \mathbf{B}\mathbf{K})$ are in the left-half plane. Since the eigenvalues of $(\mathbf{A} - \mathbf{B}\mathbf{K})$ are the regulator poles, and eigenvalues of $\mathbf{F}$ are

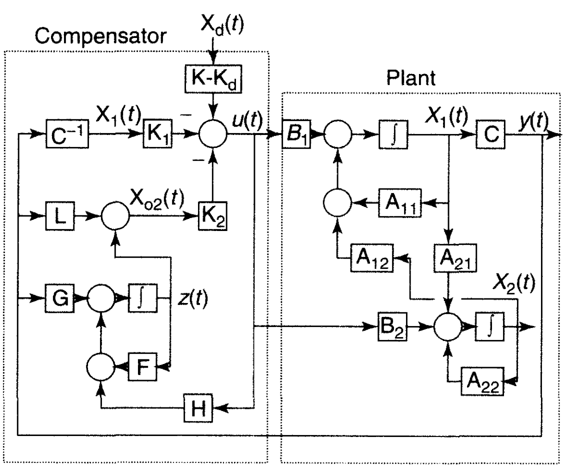

Figure 5.20 Tracking system based on reduced-order compensator

the reduced-order observer poles, it follows from Eqs. (5.136) and (5.138) that the *poles of the tracking system* are of *observer and regulator poles*. (Prove this fact by finding the eigenvalues of the closed-loop system whose state-vector is $[\mathbf{e}(t)^T; \quad \mathbf{e}_{o2}(t)^T]^T$.) According to the separation principle, the design of regulator and observer can be carried out separately by pole-placement. Note from Eqs. (5.136) and (5.138) that the order of the reduced-order tracking system is $(2n - k)$, where k is the number of measurable state-variables. Recall form the previous sub-section that the order of the full-order tracking system was $2n$. Thus, the more state-variables we can measure, the smaller will be the order of the tracking system based on reduced-order observer.

Example 5.20

Let us re-design the tracking system for the inverted pendulum on a moving cart (Example 5.18), using a reduced-order observer. Recall that it is desired to move the cart by 1 m, while not letting the pendulum fall. We have already designed a reduced-order observer for this plant in Example 5.19, using the measurement of the cart's displacement, $x(t)$, such that the observer poles are $s = -20$, $s = -20 \pm 20i$. In Example 5.18, we were able to make $(\mathbf{A}_\mathrm{d} - \mathbf{A} + \mathbf{BK}_\mathrm{d})\mathbf{x}_\mathrm{d}(t) = \mathbf{0}$ with $\mathbf{K}_\mathrm{d} = \mathbf{0}$. It remains to select the regulator gain matrix, $\mathbf{K}$, such that the eigenvalues of $(\mathbf{A} - \mathbf{BK})$ are at desired locations in the left-half plane. As in Example 5.18, let us choose the regulator poles to be $s = -0.7853 \pm 3.25328i$ and $s = -0.7853 \pm 0.7853i$. Note that we cannot directly use the regulator gain matrix of Example 5.18, because the state-vector has been *re-defined* in Example 5.19 to be $\mathbf{x}(t) = [x(t); \quad \theta(t); \quad \theta^{(1)}(t); \quad x^{(1)}(t)]^T$, as opposed to $\mathbf{x}(t) = [\theta(t); x(t); \theta^{(1)}(t); x^{(1)}(t)]^T$ of Example 5.18. The new regulator gain matrix would thus be obtained by *switching* the *first* and *second* elements of $\mathbf{K}$ calculated in Example 5.18, or by repeating pole-placement using the re-arranged state coefficient matrices as follows:

```
>>A = [A11 A12; A21 A22]; B = [B1; B2]; <enter>

>>v=[-0.7853+3.25328i -0.7853-3.25328i -0.7853+0.7853i
  -0.7853-0.7853i]'; K=place(A,B,v) <enter>

place: ndigits= 16

K =

-1.4097e+000   -2.7090e+001   -5.1339e+000   -1.9927e+000
```

The partitioning of $\mathbf{K}$ results in $\mathbf{K}_1 = -1.4097$ and $\mathbf{K}_2 = [-27.090; -5.1339; -1.9927]$. The closed-loop error dynamics matrix, $\mathbf{A}_\mathbf{CL}$, is the state-dynamics matrix obtained by combining Eqs. (5.136) and (5.138) into a state-equation, with the state-vector, $[\mathbf{e}(t); \quad \mathbf{e}_{o2}(t)]^T$, and is calculated as follows:

```
>>K2 = K(2:4); ACL = [A-B*K -B*K2; zeros(3,4) F]; <enter>
```

The eigenvalues of $\mathbf{A}_{CL}$ are calculated as follows:

```
>>eig(ACL) <enter>

ans =
-7.8530e-001+3.2533e+000i
-7.8530e-001-3.2533e+000i
-7.8530e-001+7.8530e-001i
-7.8530e-001-7.8530e-001i
-2.0000e+001+2.0000e+001i
-2.0000e+001-2.0000e+001i
-2.0000e+001
```

Note that the closed-loop eigenvalues consist of the regulator and observer poles, as expected. The *closed-loop error response* (i.e. the solution of Eqs. (5.136) and (5.138)) to $\mathbf{x}_d(t) = [1; 0; 0; 0]^T$ is nothing else but the initial response to $[\mathbf{e}(0)^T; \mathbf{e}_{o2}(0)^T]^T = [1; 0; 0; 0; 0; 0; 0]^T$, which is computed as follows:

```
>>sysCL=ss(ACL,zeros(7,1),eye(7),zeros(7,1));
  [y,t,e]= initial(sysCL,[1 zeros(1,6)]); <enter>
```

The estimation error vector, $\mathbf{e}_{o2}(t)$ is *identically zero* for this example, while the tracking errors, i.e. elements of $\mathbf{e}(t)$, are plotted in Figure 5.21. Note in Figure 5.21 that all the error transients decay to zero in about 7 s. The maximum value for the cart's velocity, $x^{(1)}(t)$, is seen to be about 0.9 m/s, while the angular velocity of the pendulum, $\theta^{(1)}(t)$, reaches a maximum value of 0.25 rad/s. The angular displacement of the pendulum, $\theta(t)$, is always less than 0.1 rad (5.73°) in magnitude, which is acceptably small for the validity of the linear plant model.

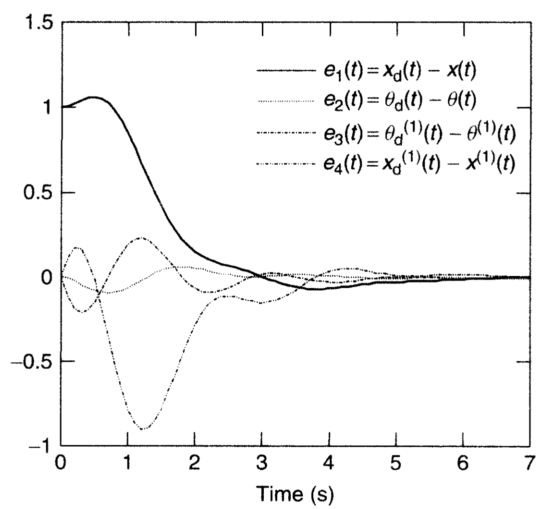

Figure 5.21 Tracking error response of closed-loop system consisting of an inverted pendulum on a moving cart and a reduced-order compensator (Example 5.20)

5.4.3 Noise and robustness issues

If noise is present in the plant, the plant's state-space representation is given by Eqs. (5.26) and (5.27), and the feedback control-law, Eq. (5.109), is modified as follows:

$$\mathbf{u}(t) = \mathbf{K}[\mathbf{x}_\mathrm{d}(t) - \mathbf{x}_\mathrm{o}(t)] - \mathbf{K}_\mathrm{d}\mathbf{x}_\mathrm{d}(t) - \mathbf{K}_\mathbf{n}\mathbf{x}_\mathbf{n}(t) \tag{5.139}$$

where $\mathbf{x}_\mathrm{d}(t)$ and $\mathbf{x}_\mathbf{n}(t)$ are the desired state-vector and the noise vector, respectively, and $\mathbf{K}_\mathrm{d}$ and $\mathbf{K}_\mathrm{n}$ are the feedforward gain matrices. In Eq. (5.139) it is assumed that both $\mathbf{x}_\mathrm{d}(t)$ and $\mathbf{x}_\mathbf{n}(t)$ can be measured, and thus need not be estimated by the observer. In case $\mathbf{x}_\mathrm{d}(t)$ and $\mathbf{x}_\mathbf{n}(t)$ are unmeasurable, we have to know the state-space model of the processes by which they are generated, in order to obtain their estimates. While it may be possible to know the dynamics of the desired state-vector, $\mathbf{x}_\mathrm{d}(t)$, the noise-vector, $\mathbf{x}_\mathbf{n}(t)$, is usually generated by a non-deterministic process whose mathematical model is unknown. In Chapter 7, we will derive observers which include an approximate model for the stochastic processes that generate noise, and design compensators for such plants. In Chapter 7 we will also study the robustness of multivariable control systems with respect to random noise.

SIMULINK can be used to simulate the response of a control system to noise, parameter variations, and nonlinearities, thereby giving a direct information about a system's robustness.

Example 5.21

Let us simulate the inverted-pendulum on a moving cart with the control system designed in Example 5.18 with a full-order compensator, with the addition of *measurement noise* modeled as a *band limited white noise* source block of SIMULINK. *White noise* is a statistical model of a special random process that we will discuss in Chapter 7. The parameter *power* of the white noise block representing the intensity of the noise is selected as 10^{-8}. A SIMULINK block-diagram of the plant with full-order compensator with regulator and observer gains designed in Example 5.18 is shown in Figure 5.22. Note the use of *matrix gain* blocks to synthesize the compensator, and a masked *subsystem* block for the state-space model of the plant. The matrix gain blocks are named B = B1 for the matrix **B**, C = C1 for the matrix **C**, L = L1 for the observer gain matrix **L**, and K = K1 = K2 for the regulator gain matrix, **K**. The *scope* outputs *thet, x, thdot* and *xdot* are the state-variables $\theta(t)$, $x(t)$, $\theta^{(1)}(t)$, and $x^{(1)}(t)$, respectively, which are *demux-ed* from the state vector of the plant, and are also saved as variables in the MATLAB workspace. The resulting simulation of $\theta(t)$ and $x(t)$ is also shown in Figure 5.22. Note the random fluctuations in both $\theta(t)$ and $x(t)$ about the desired steady-state values of $x_\mathrm{d} = 1$ m and $\theta_\mathrm{d} = 0$. The maximum magnitude of $\theta(t)$ is limited to 0.1 rad., which is within the range required for linearizing the equations of motion. However, if the intensity of the measurement noise is increased, the $\theta(t)$ oscillations quickly surpass the linear range. The simulation of Figure 5.22 also conforms to our usual experience in trying to balance a stick vertically on a finger.