6

Linear Optimal Control

6.1 The Optimal Control Problem

After designing control systems by pole-placement in Chapter 5, we naturally ask why we should need to go any further. Recall that in Chapter 5 we were faced with an *over-abundance* of design parameters for multi-input, multi-output systems. For such systems, we did not quite know how to determine all the design parameters, because only a limited number of them could be found from the closed-loop pole locations. The MATLAB M-file *place.m* imposes *additional conditions* (apart from closed-loop pole locations) to determine the design parameters for multi-input regulators, or multi-output observers; thus the design obtained by *place.m* cannot be regarded as pole-placement alone. *Optimal control* provides an alternative design strategy by which *all* the control design parameters can be determined even for multi-input, multi-output systems. Also in Chapter 5, we did not know *a priori* which pole locations would produce the desired performance; hence, some trial and error with pole locations was required before a satisfactory performance could be achieved. *Optimal control* allows us to directly formulate the performance objectives of a control system (provided we know how to do so). More importantly – apart from the above advantages – *optimal control* produces the *best possible* control system for a given set of performance objectives. What do we mean by the adjective *optimal*? The answer lies in the fact that there are many ways of doing a particular thing, but only one way which requires the *least effort*, which implies the least expenditure of *energy* (or money). For example, we can hire the most expensive lawyer in town to deal with our inconsiderate neighbor, or we can directly talk to the neighbor to achieve the desired result. Similarly, a control system can be designed to meet the desired performance objectives with the *smallest control energy*, i.e. the energy associated with generating the control inputs. Such a control system which *minimizes the cost* associated with generating control inputs is called an *optimal control system.* In contrast to the pole-placement approach, where the desired performance is *indirectly* achieved through the location of closed-loop poles, the optimal control system *directly addresses* the desired performance objectives, while minimizing the control energy. This is done by formulating an *objective function* which must be *minimized* in the design process. However, one must know how the performance objectives can be precisely translated into the objective function, which usually requires some experience with a given system.

If we define a system's *transient energy* as the total energy of the system when it is undergoing the transient response, then a successful control system must have a transient energy which quickly decays to zero. The maximum value of the transient energy indicates the maximum overshoot, while the time taken by the transient energy to decay to zero indicates the settling time. By including the transient energy in the *objective function*, we can specify the values of the acceptable maximum overshoot and settling time. Similarly, the *control energy* must also be a part of the objective function that is to be minimized. It is clear that the total control energy and total transient energy can be found by integrating the control energy and transient energy, respectively, with respect to time. Therefore, the objective function for the optimal control problem must be a *time integral* of the sum of transient energy and control energy expressed as functions of time.

6.1.1 The general optimal control formulation for regulators

Consider a linear plant described by the following state-equation:

$$\mathbf{x}^{(1)}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \tag{6.1}$$

Note that we have deliberately chosen a *time-varying* plant in Eq. (6.1), because the optimal control problem is generally formulated for time-varying systems. For simplicity, suppose we would like to design a full-state feedback regulator for the plant described by Eq. (6.1) such that the control input vector is given by

$$\mathbf{u}(t) = -\mathbf{K}(t)\mathbf{x}(t) \tag{6.2}$$

The control law given by Eq. (6.2) is linear. Since the plant is also linear, the closed-loop control system would be linear. The *control energy* can be expressed as $\mathbf{u}^{\mathrm{T}}(t)\mathbf{R}(t)\mathbf{u}(t)$, where $\mathbf{R}(t)$ is a *square, symmetric matrix* called the *control cost matrix*. Such an expression for control energy is called a *quadratic form*, because the *scalar* function, $\mathbf{u}^{\mathrm{T}}(t)\mathbf{R}(t)\mathbf{u}(t)$, contains quadratic functions of the elements of $\mathbf{u}(t)$. Similarly, the *transient energy* can also be expressed in a quadratic form as $\mathbf{x}^{\mathrm{T}}(t)\mathbf{Q}(t)\mathbf{x}(t)$, where $\mathbf{Q}(t)$ is a *square, symmetric matrix* called the *state weighting matrix*. The *objective function* can then be written as follows:

$$J(t, t_f) = \int_t^{t_f} [\mathbf{x}^{\mathrm{T}}(\tau)\mathbf{Q}(\tau)\mathbf{x}(\tau) + \mathbf{u}^{\mathrm{T}}(\tau)\mathbf{R}(\tau)\mathbf{u}(\tau)]\, d\tau \tag{6.3}$$

where t and t_f are the *initial* and *final* times, respectively, for the control to be exercised, i.e. the control *begins* at $\tau = t$ and *ends* at $\tau = t_f$, where τ is the variable of integration. The optimal control problem consists of solving for the feedback gain matrix, $\mathbf{K}(t)$, such that the scalar objective function, $J(t, t_f)$, given by Eq. (6.3) is *minimized*. However, the minimization must be carried out in such a manner that the state-vector, $\mathbf{x}(t)$, is the solution of the plant's state-equation (Eq. (6.1)). Equation (6.1) is called a *constraint* (because in its absence, $\mathbf{x}(t)$ would be free to assume *any* value), and the resulting minimization is said to be a *constrained minimization*. Hence, we are looking for a regulator gain matrix, $\mathbf{K}(t)$, which *minimizes* $J(t, t_f)$ *subject to the constraint* given by Eq. (6.1). Note that the transient term, $\mathbf{x}^{\mathrm{T}}(\tau)\mathbf{Q}(\tau)\mathbf{x}(\tau)$, in the objective function implies that a *departure* of the system's state, $\mathbf{x}(\tau)$, from the *final desired state*, $\mathbf{x}(t_f) = \mathbf{0}$, is to be minimized. In other words, the design objective is to bring $\mathbf{x}(\tau)$ to a *constant value of zero* at final time, $\tau = t_f$. If the final desired state is *non-zero*, the objective function can be modified appropriately, as we will see later.

By substituting Eq. (6.2) into Eq. (6.1), the closed-loop state-equation can be written as follows:

$$\mathbf{x}^{(1)}(t) = [\mathbf{A}(t) - \mathbf{B}(t)\mathbf{K}(t)]\mathbf{x}(t) = \mathbf{A}_{\mathbf{CL}}(t)\mathbf{x}(t) \tag{6.4}$$

where $\mathbf{A}_{\mathbf{CL}}(t) = [\mathbf{A}(t) - \mathbf{B}(t)\mathbf{K}(t)]$, the closed-loop state-dynamics matrix. The solution to Eq. (6.4) can be written as follows:

$$\mathbf{x}(t) = \Phi_{\mathbf{CL}}(t, t_0)\mathbf{x}(t_0) \tag{6.5}$$

where $\Phi_{\mathbf{CL}}(t, t_0)$ is the *state-transition matrix* of the time-varying closed-loop system represented by Eq. (6.4). Since the system is time-varying, $\Phi_{\mathbf{CL}}(t, t_0)$, is *not* the matrix exponential of $\mathbf{A}_{\mathbf{CL}}(t - t_0)$, but is related in *some other way* (which we do not know) to $\mathbf{A}_{\mathbf{CL}}(t)$. Equation (6.5) indicates that the state at any time, $\mathbf{x}(t)$, can be obtained by post-multiplying the state at some initial time, $\mathbf{x}(t_0)$, with $\Phi_{\mathbf{CL}}(t, t_0)$. On substituting Eq. (6.5) into Eq. (6.3), we get the following expression for the objective function:

$$J(t, t_f) = \int_t^{t_f} \mathbf{x}^{\mathbf{T}}(t)\Phi^{\mathbf{T}}_{\mathbf{CL}}(\tau, t)[\mathbf{Q}(\tau) + \mathbf{K}^{\mathbf{T}}(\tau)\mathbf{R}(\tau)\mathbf{K}(\tau)]\Phi_{\mathbf{CL}}(\tau, t)\mathbf{x}(t)\, d\tau \tag{6.6}$$

or, taking the *initial state-vector*, $\mathbf{x}(t)$, outside the integral sign, we can write

$$J(t, t_f) = \mathbf{x}^{\mathbf{T}}(t)\mathbf{M}(t, t_f)\mathbf{x}(t) \tag{6.7}$$

where

$$\mathbf{M}(t, t_f) = \int_t^{t_f} \Phi^{\mathbf{T}}_{\mathbf{CL}}(\tau, t)[\mathbf{Q}(\tau) + \mathbf{K}^{\mathbf{T}}(\tau)\mathbf{R}(\tau)\mathbf{K}(\tau)]\Phi_{\mathbf{CL}}(\tau, t)\, d\tau \tag{6.8}$$

Equation (6.7) shows that the objective function is a *quadratic function* of the initial state, $\mathbf{x}(t)$. Hence, the linear optimal regulator problem posed by Eqs. (6.1)–(6.3) is also called the *linear, quadratic regulator* (LQR) problem. You can easily show from Eq. (6.8) that $\mathbf{M}(t, t_f)$ is a symmetric matrix, i.e. $\mathbf{M}^{\mathrm{T}}(t, t_f) = \mathbf{M}(t, t_f)$, because both $\mathbf{Q}(t)$ and $\mathbf{R}(t)$ are symmetric. On substituting Eq. (6.5) into Eq. (6.6), we can write the objective function as follows:

$$J(t, t_f) = \int_t^{t_f} \mathbf{x}^{\mathrm{T}}(\tau)[\mathbf{Q}(\tau) + \mathbf{K}^{\mathbf{T}}(\tau)\mathbf{R}(\tau)\mathbf{K}(\tau)]\mathbf{x}(\tau)\, d\tau \tag{6.9}$$

On differentiating Eq. (6.9) *partially* with respect to the lower limit of integration, t, according to the *Leibniz rule* (see a textbook on integral calculus, such as that by Kreyszig [1]), we get the following:

$$\partial J(t, t_f)/\partial t = -\mathbf{x}^{\mathbf{T}}(t)[\mathbf{Q}(t) + \mathbf{K}^{\mathrm{T}}(t)\mathbf{R}(t)\mathbf{K}(t)]\mathbf{x}(t) \tag{6.10}$$

where ∂ denotes *partial differentiation*. Also, partial differentiation of Eq. (6.7) with respect to t results in the following:

$$\partial J(t, t_f)/\partial t = [\mathbf{x}^{(1)}(t)]^T\mathbf{M}(t, t_f)\mathbf{x}(t) + \mathbf{x}^{\mathrm{T}}(t)[\partial\mathbf{M}(t, t_f)/\partial t]\mathbf{x}(t) + \mathbf{x}^{\mathbf{T}}(t)\mathbf{M}(t, t_f)\mathbf{x}^{(1)}(t) \tag{6.11}$$

On substituting $\mathbf{x}^{(1)}(t) = \mathbf{A}_{\mathrm{CL}}(t)\mathbf{x}(t)$ from Eq. (6.4) into Eq. (6.11), we can write

$$\partial J(t, t_f)/\partial t = \mathbf{x}^{\mathrm{T}}(t)[\mathbf{A}_{\mathrm{CL}}^{\mathrm{T}}(t)\mathbf{M}(t, t_f) + \partial\mathbf{M}(t, t_f)/\partial t + \mathbf{M}(t, t_f)\mathbf{A}_{\mathrm{CL}}(t)]\mathbf{x}(t) \quad (6.12)$$

Equations (6.10) and (6.12) are *quadratic forms* for the *same* scalar function, $\partial J(t, t_f)/\partial t$ in terms of the *initial state*, $\mathbf{x}(t)$. Equating Eqs. (6.10) and (6.12), we get the following *matrix differential equation* to be satisfied by $\mathbf{M}(t, t_f)$:

$$-[\mathbf{Q}(t) + \mathbf{K}^{\mathrm{T}}(t)\mathbf{R}(t)\mathbf{K}(t)] = \mathbf{A}_{\mathrm{CL}}^{\mathrm{T}}(t)\mathbf{M}(t, t_f) + \partial\mathbf{M}(t, t_f)/\partial t + \mathbf{M}(t, t_f)\mathbf{A}_{\mathrm{CL}}(t) \quad (6.13)$$

or

$$-\partial\mathbf{M}(t, t_f)/\partial t = \mathbf{A}_{\mathrm{CL}}^{\mathrm{T}}(t)\mathbf{M}(t, t_f) + \mathbf{M}(t, t_f)\mathbf{A}_{\mathrm{CL}}(t) + [\mathbf{Q}(t) + \mathbf{K}^{\mathrm{T}}(t)\mathbf{R}(t)\mathbf{K}(t)] \quad (6.14)$$

Equation (6.14) is a first order, *matrix partial differential equation* in terms of the initial time, t, whose solution $\mathbf{M}(t, t_f)$ is given by Eq. (6.8). However, since we do not know the state transition matrix, $\Phi_{\mathrm{CL}}(\tau, t)$, of the general time-varying, closed-loop system, Eq. (6.8) is useless to us for determining $\mathbf{M}(t, t_f)$. Hence, the *only way* to find the unknown matrix $\mathbf{M}(t, t_f)$ is by solving the *matrix differential equation*, Eq. (6.14). We need *only one initial condition* to solve the *first order* matrix differential equation, Eq. (6.14). The simplest initial condition can be obtained by putting $t = t_f$ in Eq. (6.8), resulting in

$$\mathbf{M}(t_f, t_f) = \mathbf{0} \quad (6.15)$$

The linear optimal control problem is thus posed as finding the *optimal regulator gain matrix*, $\mathbf{K}(t)$, such that the solution, $\mathbf{M}(t, t_f)$, to Eq. (6.14) (and hence the objective function, $J(t, t_f)$) is *minimized*, subject to the *initial condition*, Eq. (6.15). The choice of the matrices $\mathbf{Q}(t)$ and $\mathbf{R}(t)$ is left to the designer. However, as we will see below, these two matrices specifying performance objectives and control effort, cannot be arbitrary, but must obey certain conditions.

6.1.2 Optimal regulator gain matrix and the Riccati equation

Let us denote the *optimal feedback gain matrix* that *minimizes* $\mathbf{M}(t, t_f)$ by $\mathbf{K}_o(t)$. The *minimum* value of $\mathbf{M}(t, t_f)$ which results from the optimal gain matrix, $\mathbf{K}_o(t)$, is denoted by $\mathbf{M}_o(t, t_f)$, and the minimum value of the objective function is denoted by $J_o(t, t_f)$. For simplicity of notation, let us drop the functional arguments for the time being, and denote $\mathbf{M}(t, t_f)$ by $\mathbf{M}$, $J(t, t_f)$ by J, etc. Then, according to Eq. (6.7), the minimum value of the objective function is the following:

$$J_o = \mathbf{x}^T(t)\mathbf{M}_o\mathbf{x}(t) \quad (6.16)$$

Since J_o is the minimum value of J for any initial state, $\mathbf{x}(t)$, we can write $J_o \leq J$, or

$$\mathbf{x}^T(t)\mathbf{M}_o\mathbf{x}(t) \leq \mathbf{x}^T(t)\mathbf{M}\mathbf{x}(t) \quad (6.17)$$

If we express $\mathbf{M}$ as follows:

$$\mathbf{M} = \mathbf{M}_o + \mathbf{m} \tag{6.18}$$

and substitute Eq. (6.18) into Eq. (6.17), the following condition must be satisfied:

$$\mathbf{x}^T(t)\mathbf{M}_o\mathbf{x}(t) \leq \mathbf{x}^T(t)\mathbf{M}_o\mathbf{x}(t) + \mathbf{x}^T(t)\mathbf{m}\mathbf{x}(t) \tag{6.19}$$

or

$$\mathbf{x}^T(t)\mathbf{m}\mathbf{x}(t) \geq 0 \tag{6.20}$$

A matrix, $\mathbf{m}$, which satisfies Eq. (6.20), is called a *positive semi-definite* matrix. Since $\mathbf{x}(t)$ is an *arbitrary* initial state-vector, you can show that according to Eq. (6.20), *all eigenvalues of* $\mathbf{m}$ must be *greater than or equal to zero*.

It now remains to derive an expression for the optimum regulator gain matrix, $\mathbf{K}_o(t)$, such that $\mathbf{M}$ is minimized. If $\mathbf{M}_o$ is the minimum value of $\mathbf{M}$, then $\mathbf{M}_o$ must satisfy Eq. (6.14) when $\mathbf{K}(t) = \mathbf{K}_o(t)$, i.e.

$$-\partial\mathbf{M}_o/\partial t = \mathbf{A}_{\mathbf{CL}}^{\mathbf{T}}(t)\mathbf{M}_o + \mathbf{M}_o\mathbf{A}_{\mathbf{CL}}(t) + [\mathbf{Q}(t) + \mathbf{K}_o^{\mathbf{T}}(t)\mathbf{R}(t)\mathbf{K}_o(t)] \tag{6.21}$$

Let us express the gain matrix, $\mathbf{K}(t)$, in terms of the optimal gain matrix, $\mathbf{K}_o(t)$, as follows:

$$\mathbf{K}(t) = \mathbf{K}_o(t) + \mathbf{k}(t) \tag{6.22}$$

On substituting Eqs. (6.18) and (6.21) into Eq. (6.14), we can write

$$\begin{aligned}-\partial(\mathbf{M}_o + \mathbf{m})/\partial t &= \mathbf{A}_{\mathbf{CL}}^{\mathbf{T}}(t)(\mathbf{M}_o + \mathbf{m}) + (\mathbf{M}_o + \mathbf{m})\mathbf{A}_{\mathbf{CL}}(t) \\ &\quad + [\mathbf{Q}(t) + \{\mathbf{K}_o(t) + \mathbf{k}(t)\}^{\mathbf{T}}\mathbf{R}(t)\{\mathbf{K}_o(t) + \mathbf{k}(t)\}]\end{aligned} \tag{6.23}$$

On subtracting Eq. (6.21) from Eq. (6.23), we get

$$-\partial\mathbf{m}/\partial t = \mathbf{A}_{\mathbf{CL}}^{\mathbf{T}}(t)\mathbf{m} + \mathbf{m}\mathbf{A}_{\mathbf{CL}}(t) + \mathbf{S} \tag{6.24}$$

where

$$\mathbf{S} = [\mathbf{K}_o^{\mathbf{T}}(t)\mathbf{R}(t) - \mathbf{M}_o\mathbf{B}(t)]\mathbf{k}(t) + \mathbf{k}^{\mathbf{T}}(t)[\mathbf{R}(t)\mathbf{K}_o(t) - \mathbf{B}^{\mathbf{T}}(t)\mathbf{M}_o] + \mathbf{k}^{\mathbf{T}}(t)\mathbf{R}(t)\mathbf{k}(t) \tag{6.25}$$

Comparing Eq. (6.24) with Eq. (6.14), we find that the two equations are of the *same form*, with the term $[\mathbf{Q}(t) + \mathbf{K}^{\mathbf{T}}(t)\mathbf{R}(t)\mathbf{K}(t)]$ in Eq. (6.14) replaced by $\mathbf{S}$ in Eq. (6.24). Since the non-optimal matrix, $\mathbf{M}$, in Eq. (6.14) satisfies Eq. (6.8), it must be true that $\mathbf{m}$ satisfies the following equation:

$$\mathbf{m}(t, t_f) = \int_t^{t_f} \Phi_{\mathbf{CL}}^{\mathbf{T}}(\tau, t)\mathbf{S}(\tau, t_f)\Phi_{\mathbf{CL}}(\tau, t)\, d\tau \tag{6.26}$$

Recall from Eq. (6.20) that $\mathbf{m}$ must be positive semi-definite. However, Eq. (6.26) requires that for $\mathbf{m}$ to be positive semi-definite, the matrix $\mathbf{S}$ given by Eq. (6.25) must be positive