semi-definite. Looking at Eq. (6.25), we find that **S** can be positive semi-definite if and only if the *linear terms* in Eq. (6.25) are zeros, i.e. which implies

$$\mathbf{K}_o^T(t)\mathbf{R}(t) - \mathbf{M}_o\mathbf{B}(t) = \mathbf{0} \tag{6.27}$$

or the optimal feedback gain matrix is given by

$$\mathbf{K}_o(t) = \mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{M}_o \tag{6.28}$$

Substituting Eq. (6.28) into Eq. (6.21), we get the following differential equation to be satisfied by the optimal matrix, $\mathbf{M}_o$:

$$-\partial\mathbf{M}_o/\partial t = \mathbf{A}^T(t)\mathbf{M}_o + \mathbf{M}_o\mathbf{A}(t) - \mathbf{M}_o\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{M}_o + \mathbf{Q}(t) \tag{6.29}$$

Equation (6.29) has a special name: the *matrix Riccati equation*. The matrix Riccati equation is special because it's solution, $\mathbf{M}_o$, substituted into Eq. (6.28), gives us the optimal feedback gain matrix, $\mathbf{K}_o(t)$. Exact solutions to the Riccati equation are rare, and in most cases a numerical solution procedure is required. Note that Riccati equation is a first order, nonlinear differential equation, and can be solved by numerical methods similar to those discussed in Chapter 4 for solving the nonlinear state-equations, such as the *Runge–Kutta* method, or other more convenient methods (such as the one we will discuss in Section 6.5). However, in contrast to the state-equation, the solution is a matrix rather than a vector, and the solution procedure has to *march backwards in time*, since the *initial condition* for Riccati equation is specified (Eq. (6.15)) at the *final time*, $t = t_f$, as follows:

$$\mathbf{M}_o(t_f, t_f) = \mathbf{0} \tag{6.30}$$

For this reason, the condition given by Eq. (6.30) is called the *terminal condition* rather than initial condition. Note that the solution to Eq. (6.29) is $\mathbf{M}_o(t, t_f)$ where $t < t_f$. Let us defer the solution to the matrix Riccati equation until Section 6.5.

In summary, the optimal control procedure using full-state feedback consists of specifying an objective function by suitably selecting the performance and control cost weighting matrices, $\mathbf{Q}(t)$ and $\mathbf{R}(t)$, and solving the Riccati equation subject to the terminal condition, in order to determine the full-state feedback matrix, $\mathbf{K}_o(t)$. In most cases, rather than solving the general time-varying optimal control problem, certain simplifications can be made which result in an easier problem, as seen in the following sections.

6.2 Infinite-Time Linear Optimal Regulator Design

A large number of control problems are such that the control interval, $(t_f - t)$, is *infinite*. If we are interested in a specific *steady-state* behavior of the control system, we are interested in the response, $\mathbf{x}(t)$, when $t_f \to \infty$, and hence the control interval is *infinite*. The approximation of an infinite control interval results in a simplification in the optimal control problem, as we will see below. For infinite final time, the quadratic objective function can be expressed as follows:

$$J_\infty(t) = \int_t^\infty [\mathbf{x}^T(\tau)\mathbf{Q}(\tau)\mathbf{x}(\tau) + \mathbf{u}^T(\tau)\mathbf{R}(\tau)\mathbf{u}(\tau)]\, d\tau \tag{6.31}$$

where $J_\infty(t)$ indicates the objective function of the infinite final time (or steady-state) optimal control problem. For the infinite final time, the *backward time integration* of the matrix Riccati equation (Eq. (6.29)), beginning from $\mathbf{M_o}(\infty, \infty) = \mathbf{0}$, would result in a solution, $\mathbf{M_o}(t, \infty)$, which is *either* a *constant*, or *does not converge* to any limit. If the numerical solution to the Riccati equation converges to a constant value, then $\partial\mathbf{M_o}/\partial t = \mathbf{0}$, and the Riccati equation becomes

$$\mathbf{0} = \mathbf{A}^T(t)\mathbf{M_o} + \mathbf{M_o}\mathbf{A}(t) - \mathbf{M_o}\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{M_o} + \mathbf{Q}(t) \tag{6.32}$$

Note that Eq. (6.32) is no longer a differential equation, but an *algebraic equation*. Hence, Eq. (6.32) is called the *algebraic Riccati equation*. The feedback gain matrix is given by Eq. (6.28), in which $\mathbf{M_o}$ is the (constant) solution to the algebraic Riccati equation. It is (relatively) much easier to solve Eq. (6.32) rather that Eq. (6.29). However, a solution to the algebraic Riccati equation *may not* always exist.

What are the conditions for the existence of the positive semi-definite solution to the algebraic Riccati equation? This question is best answered in a textbook devoted to optimal control, such as that by Bryson and Ho [2], and involves precise mathematical conditions, such as *stabilizability*, *detectability*, etc., for the existence of solution. Here, it suffices to say that for all practical purposes, if *either* the plant is *asymptotically stable*, *or* the plant is *controllable* and *observable* with the output, $\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t)$, where $\mathbf{C}^T(t)\mathbf{C}(t) = \mathbf{Q}(t)$, and $\mathbf{R}(t)$ is a symmetric, *positive definite* matrix, then there is a *unique, positive definite* solution, $\mathbf{M_o}$, to the algebraic Riccati equation. Note that $\mathbf{C}^T(t)\mathbf{C}(t) = \mathbf{Q}(t)$ implies that $\mathbf{Q}(t)$ must be a *symmetric* and *positive semi-definite* matrix. Furthermore, the requirement that the control cost matrix, $\mathbf{R}(t)$, must be *symmetric* and *positive definite* (i.e. *all eigenvalues* of $\mathbf{R}(t)$ must be *positive* real numbers) for the solution, $\mathbf{M_o}$ to be *positive definite* is clear from Eq. (6.25), which implies that $\mathbf{S}$ (and hence $\mathbf{m}$) will be positive definite only if $\mathbf{R}(t)$ is positive definite. Note that these are *sufficient* (but not *necessary*) conditions for the existence of a unique solution to the algebraic Riccati equation, i.e. there may be plants that *do not* satisfy these conditions, and yet there *may exist* a unique, positive definite solution for such plants. A less restrictive set of sufficient conditions for the existence of a unique, positive definite solution to the algebraic Riccati equation is that the plant must be *stabilizable* and *detectable* with the output, $\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t)$, where $\mathbf{C}^T(t)\mathbf{C}(t) = \mathbf{Q}(t)$, and $\mathbf{R}(t)$ is a symmetric, *positive definite* matrix (see Bryson and Ho [2] for details).

While Eq. (6.32) has been derived for linear optimal control of time-varying plants, its usual application is to *time-invariant* plants, for which the algebraic Riccati equation is written as follows:

$$\mathbf{0} = \mathbf{A}^T\mathbf{M_o} + \mathbf{M_o}\mathbf{A} - \mathbf{M_o}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{M_o} + \mathbf{Q} \tag{6.33}$$

In Eq. (6.33), all the matrices are *constant* matrices. MATLAB contains a solver for the algebraic Riccati equation for time-invariant plants in the M-file named *are.m*. The command *are* is used as follows:

```
>>x = are(a,b,c) <enter>
```

where $\mathbf{a} = \mathbf{A}$, $\mathbf{b} = \mathbf{BR}^{-1}\mathbf{B}^T$, $\mathbf{c} = \mathbf{Q}$, in Eq. (6.33), and the returned solution is $\mathbf{x} = \mathbf{M}_o$. For the existence of a unique, positive definite solution to Eq. (6.33), the sufficient conditions remains the same, i.e. the plant with coefficient matrices **A**, **B** must be controllable, **Q** must be symmetric and positive semi-definite, and **R** must be symmetric and positive definite. Another MATLAB function, *ric*, computes the error in solving the algebraic Riccati equation. Alternatively, MATLAB's Control System Toolbox (CST) provides the functions *lqr* and *lqr2* for the solution of the linear optimal control problem with a quadratic objective function, using two different numerical schemes. The command *lqr* (or *lqr2*) is used as follows:

```
>>[Ko,Mo,E]= lqr(A,B,Q,R) <enter>
```

where **A**, **B**, **Q**, **R** are the same as in Eq. (6.33), $\mathbf{Mo} = \mathbf{M_o}$, the returned solution of Eq. (6.33), $\mathbf{Ko} = \mathbf{K}_o = \mathbf{R}^{-1}\mathbf{B}^T\mathbf{M}_o$ the returned optimal regulator gain matrix, and **E** is the vector containing the closed-loop eigenvalues (i.e. the eigenvalues of $\mathbf{A}_{CL} = \mathbf{A} - \mathbf{BK_o}$). The command *lqr* (or *lqr2*) is more convenient to use, since it directly works with the plant's coefficient matrices and the weighting matrices. Let us consider a few examples of linear optimal control of time-invariant plants, based upon the solution of the algebraic Riccati equation. (For time-varying plants, the optimal feedback gain matrix can be determined by solving the algebraic Riccati equation at each instant of time, t, using either *lqr* or *lqr2* in a time-marching procedure.)

Example 6.1

Consider the longitudinal motion of a flexible bomber aircraft of Example 4.7. The sixth order, two input system is described by the linear, time-invariant, state-space representation given by Eq. (4.71). The inputs are the *desired elevator deflection* (rad.), $u_1(t)$, and the *desired canard deflection* (rad.), $u_2(t)$, while the outputs are the *normal acceleration* (m/s^2), $y_1(t)$, and the *pitch-rate* (rad./s), $y_2(t)$. Let us design an optimal regulator which would produce a maximum overshoot of less than ± 2 m/s^2 in the normal-acceleration and less than ± 0.03 rad/s in pitch-rate, and a settling time less than 5 s, while requiring elevator and canard deflections *not exceeding* ± 0.1 rad. (5.73°), if the initial condition is 0.1 rad/s perturbation in the pitch-rate, i.e. $\mathbf{x}(0) = [0; 0.1; 0; 0; 0; 0]^T$.

What **Q** and **R** matrices should we choose for this problem? Note that **Q** is a square matrix of size (6×6) and **R** is a square matrix of size (2×2). Examining the plant model given by Eq. (4.71), we find that while the normal acceleration, $y_1(t)$, depends upon all the six state variables, the pitch-rate, $y_2(t)$, is equal to the second state-variable. Since we have to enforce the maximum overshoot limits on $y_1(t)$ and $y_2(t)$, we must, therefore, impose certain limits on the maximum overshoots of all the state variables, which is done by selecting an appropriate state weighting matrix, **Q**. Similarly, the maximum overshoot limits on the two input variables, $u_1(t)$ and $u_2(t)$, must be specified through the control cost matrix, **R**. The settling time would be determined by both **Q** and **R**. *A priori*, we do not quite know what values of **Q** and **R** will produce the desired objectives. Hence, some trial and error is required in selecting the appropriate **Q** and **R**. Let us begin by selecting both **Q** and **R** as *identity matrices*. By doing so, we are specifying that all the six state variables and the two control inputs are *equally important* in the objective function, i.e. it is equally important to bring all the state variables and the control inputs to zero, while minimizing their overshoots. Note that the existence of a unique, positive definite solution to the algebraic Riccati equation will be guaranteed if **Q** and **R** are positive semi-definite and positive definite, respectively, and the plant is controllable. Let us test whether the plant is controllable as follows:

```
>>rank(ctrb(A,B)) <enter>

ans=
  6
```

Hence, the plant is controllable. By choosing $\mathbf{Q} = \mathbf{I}$, and $\mathbf{R} = \mathbf{I}$, we are ensuring that both are positive definite. Therefore, all the sufficient conditions for the existence of an optimal solution are satisfied. For solving the algebraic Riccati equation, let us use the MATLAB command *lqr* as follows:

```
>>[Ko,Mo,E]=lqr(A,B,eye(6),eye(2)) <enter>

Ko=
 3.3571e+000 -4.2509e-001 -6.2538e-001 -7.3441e-001 2.8190e+000  1.5765e+000
 3.8181e+000 1.0274e+000  -5.4727e-001 -6.8075e-001 2.1020e+000  1.8500e+000

Mo=
 1.7429e+000 2.8673e-001  1.1059e-002  -1.4159e-002 4.4761e-002  3.8181e-002
 2.8673e-001 4.1486e-001  1.0094e-002  -2.1528e-003 -5.6679e-003 1.0274e-002
 1.1059e-002 1.0094e-002  1.0053e+000  4.4217e-003  -8.3383e-003 -5.4727e-003
-1.4159e-002 -2.1528e-003 4.4217e-003  4.9047e-003  -9.7921e-003 -6.8075e-003
 4.4761e-002 -5.6679e-003 -8.3383e-003 -9.7921e-003 3.7586e-002  2.1020e-002
 3.8181e-002 1.0274e-002  -5.4727e-003 -6.8075e-003 2.1020e-002  1.8500e-002

E =
-2.2149e+002+2.0338e+002i
-2.2149e+002-2.0338e+002i
-1.2561e+002
-1.8483e+000+1.3383e+000i
-1.8483e+000-1.3383e+000i
-1.0011e+000
```

To see whether this design is acceptable, we calculate the initial response of the closed-loop system as follows:

```
>>sys1=ss(A-B*Ko,zeros(6,2),C,zeros(2,2));<enter>

>>[Y1,t1,X1]=initial(sys1,[0.1 zeros(1,5)]'); u1=-Ko*X1'; <enter>
```

Let us try another design with $\mathbf{Q} = 0.01\mathbf{I}$, and $\mathbf{R} = \mathbf{I}$. As compared with the previous design, we are now specifying that it is 100 times *more important* to minimize the total control energy than minimizing the total transient energy. The new regulator gain matrix is determined by re-solving the algebraic Riccati equation with $\mathbf{Q} = 0.01\mathbf{I}$ and $\mathbf{R} = \mathbf{I}$ as follows:

```
>>[Ko,Mo,E] = lqr(A,B,0.01*eye(6),eye(2)) <enter>

Ko=
 1.0780e+000 -1.6677e-001 -4.6948e-002 -7.5618e-002 5.9823e-001  3.5302e-001
 1.3785e+000 3.4502e-001  -1.3144e-002 -6.5260e-002 4.7069e-001  3.0941e-001

Mo=
 4.1913e-001 1.2057e-001  9.2728e-003  -2.2727e-003 1.4373e-002  1.3785e-002
 1.2057e-001 1.0336e-001  6.1906e-003  -3.9125e-004 -2.2236e-003 3.4502e-003
 9.2728e-003 6.1906e-003  1.0649e-002  9.7083e-005  -6.2597e-004 -1.3144e-004
-2.2727e-003 -3.9125e-004 9.7083e-005  1.7764e-004  -1.0082e-003 -6.5260e-004
 1.4373e-002 -2.2236e-003 -6.2597e-004 -1.0082e-003 7.9764e-003  4.7069e-003
 1.3785e-002 3.4502e-003  -1.3144e-004 -6.5260e-004 4.7069e-003  3.0941e-003

E =
-9.1803e+001
-7.8748e+001+5.0625e+001i
-7.8748e+001-5.0625e+001i
-1.1602e+000+1.7328e+000i
-1.1602e+000-1.7328e+000i
-1.0560e+000
```

The closed-loop state-space model, closed-loop initial response and the required inputs are calculated as follows:

```
>>sys2=ss(A-B*Ko,zeros(6,2),C,zeros(2,2)); <enter>

>>[Y2,t2,X2] = initial(sys2,[0.1 zeros(1,5)]'); u2=-Ko*X2'; <enter>
```

Note that the closed-loop eigenvalues (contained in the returned matrix **E**) of the first design are *further inside* the left-half plane than those of the second design, which indicates that the first design would have a *smaller* settling time, and a *larger* input requirement when compared to the second design. The resulting outputs, $y_1(t)$ and $y_2(t)$, for the two regulator designs are compared with the plant's initial response to the same initial condition in Figure 6.1.

The plant's oscillating initial response is seen in Figure 6.1 to have maximum overshoots of -20 m/s^2 and -0.06 rad/s, for $y_1(t)$ and $y_2(t)$, respectively, and a settling time exceeding 5 s (actually about 10 s). Note in Figure 6.1 that while the first design ($\mathbf{Q} = \mathbf{I}, \mathbf{R} = \mathbf{I}$) produces the closed-loop initial response of $y_2(t)$, $u_1(t)$, and $u_2(t)$ within acceptable limits, the response of $y_1(t)$ displays a maximum overshoot of 10 m/s^2 (beginning at -15 m/s^2 at $t = 0$, and shooting to -25 m/s^2), which is unacceptable. The settling time of the first design is about 3 s, while that of the second design ($\mathbf{Q} = 0.01\mathbf{I}, \mathbf{R} = \mathbf{I}$) is slightly less than 5 s. The second design produces a maximum overshoot of $y_1(t)$ less than 2 m/s^2 and that of $y_2(t)$ about -0.025 rad/s, which is acceptable. The required control inputs, $u_1(t)$ and $u_2(t)$, for the two designs are plotted in Figure 6.2. While the first design requires a maximum

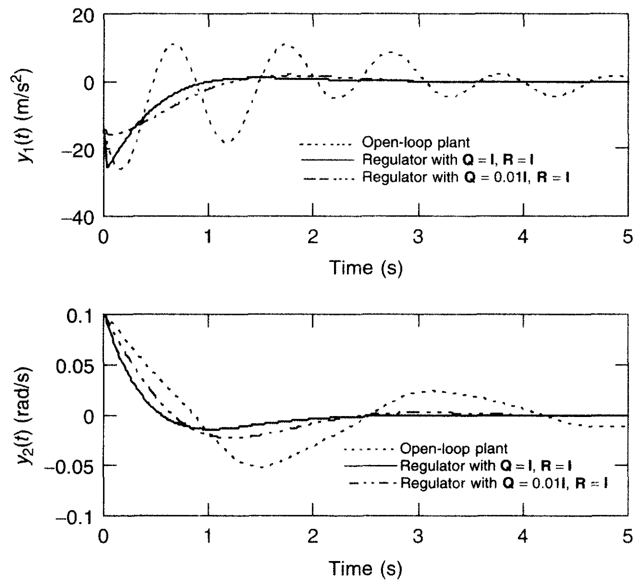

Figure 6.1 Open and closed-loop initial response of the regulated flexible bomber aircraft, for two optimal regulator designs

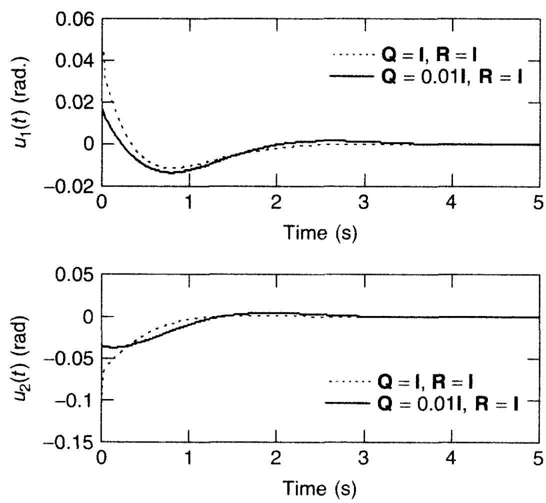

Figure 6.2 Required initial response control inputs of the regulated flexible bomber aircraft, for two optimal regulator designs

value of elevator deflection, $u_1(t)$, about 0.045 rad., the second design is seen to require a maximum value of $u_1(t)$ less than 0.02 rad. Similarly, the canard deflection, $u_2(t)$, for the second design has a smaller maximum value (-0.04 rad) than that of the first design (-0.1 rad.). Hence, the second design fulfills all the design objectives.

In Example 6.1, we find that the total transient energy is *more sensitive* to the *settling time*, than the maximum overshoot. Recall from Chapter 5 that if we try to reduce the settling time, we have to accept an increase in the maximum overshoot. Conversely, to reduce the maximum overshoot of $y_1(t)$, which depends upon all the state variables, we must allow an increase in the settling time, which is achieved in the second design by reducing the importance of minimizing the transient energy by hundred-fold, as compared to the first design. Let us now see what effect a measurement noise will have on the closed-loop initial response. We take the second regulator design (i.e. $\mathbf{Q} = 0.01\mathbf{I}$, $\mathbf{R} = \mathbf{I}$) and simulate the initial response assuming a random error (i.e. measurement noise) in feeding back the pitch-rate (the second state-variable of the plant). The simulation is carried out using SIMULINK block-diagram shown in Figure 6.3, where the measurement noise is simulated by the *band-limited white noise* block with a *power* parameter of 10^{-4}. Note the

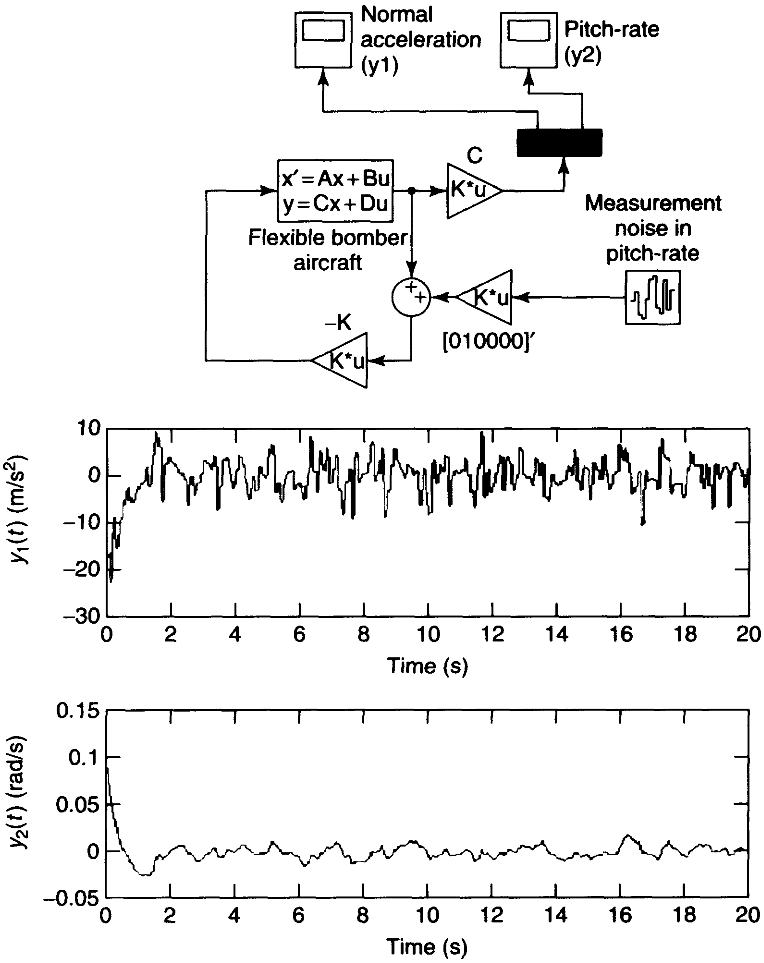

Figure 6.3 Simulation of initial response of the flexible bomber with a full-state feedback regulator and measurement noise in the pitch-rate channel

manner in which the noise is added to the feedback loop through the *matrix gain* block. The simulated initial response is also shown in Figure 6.3. Note the random fluctuations in both normal acceleration, $y_1(t)$, and pitch-rate, $y_2(t)$. The aircraft crew are likely to have a rough ride due to large sustained fluctuations (± 10 m/s^2) in normal acceleration, $y_1(t)$, resulting from the small measurement noise! The feedback loop results in an amplification of the measurement noise. If the elements of the feedback gain matrix, **K**, corresponding to pitch-rate are reduced in magnitude then the noise due to pitch-rate feedback will be alleviated. Alternatively, pitch-rate (or any other state-variable that is noisy) can be removed from state feedback, with the use of an observer based compensator that feeds back only selected state-variables (see Chapter 5).

Example 6.2

Let us design an optimal regulator for the flexible, rotating spacecraft shown in Figure 6.4. The spacecraft consists of a *rigid hub* and four *flexible appendages*, each having a *tip mass*, with three *torque* inputs, $u_1(t)$, $u_2(t)$, $u_3(t)$, and three *angular rotation* outputs in rad., $y_1(t)$, $y_2(t)$, $y_3(t)$. Due to the flexibility of the appendages, the spacecraft is a *distributed parameter* system (see Chapter 1). However, it is approximated by a *lumped parameter*, linear, time-invariant state-space representation using a *finite-element model* [3]. The order of the spacecraft can be reduced to 26 for accuracy in a desired frequency range [4]. The 26th order state-vector, $\mathbf{x}(t)$, of the spacecraft consists of the angular displacement, $y_1(t)$, and *angular velocity* of the rigid hub, combined with individual *transverse* (i.e. *perpendicular* to the appendage) *displacements* and *transverse velocities* of three points on each appendage. The state-coefficient matrices of the spacecraft are given as follows:

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & \mathbf{0} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\mathbf{d} \end{bmatrix} \quad \mathbf{C} = [\,\mathbf{d}^{\mathrm{T}};\ \ \mathbf{0}\,]; \quad \mathbf{D} = \mathbf{0} \tag{6.34}$$

where **M, K**, and **d** are the *mass*, *stiffness*, and *control influence* matrices, given in Appendix C.

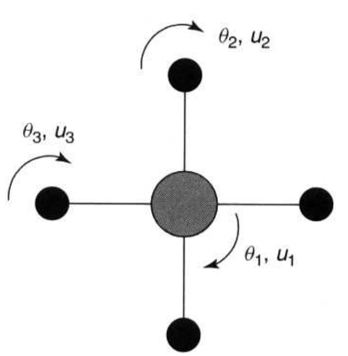

Figure 6.4 A rotating, flexible spacecraft with three inputs, (u_1, u_2, u_3), and three outputs $(\theta_1, \theta_2, \theta_3)$

The eigenvalues of the spacecraft are the following:

```
>>damp(A) <enter>

Eigenvalue                    Damping       Freq.(rad/sec)
9.4299e-013+7.6163e+003i     -6.1230e-017   7.6163e+003
9.4299e-013-7.6163e+003i     -6.1230e-017   7.6163e+003
8.1712e-013+4.7565e+004i     -6.1230e-017   4.7565e+004
8.1712e-013-4.7565e+004i     -6.1230e-017   4.7565e+004
8.0539e-013+2.5366e+003i     -2.8327e-016   2.5366e+003
8.0539e-013-2.5366e+003i     -2.8327e-016   2.5366e+003
7.4867e-013+4.7588e+004i     -6.1230e-017   4.7588e+004
7.4867e-013-4.7588e+004i     -6.1230e-017   4.7588e+004
7.1276e-013+2.5982e+004i     -6.1230e-017   2.5982e+004
7.1276e-013-2.5982e+004i     -6.1230e-017   2.5982e+004
5.2054e-013+1.4871e+004i     -6.1230e-017   1.4871e+004
5.2054e-013-1.4871e+004i     -6.1230e-017   1.4871e+004
4.8110e-013+2.0986e+002i     -2.2817e-015   2.0986e+002
4.8110e-013-2.0986e+002i     -2.2817e-015   2.0986e+002
4.4812e-013+2.6009e+004i     -6.1230e-017   2.6009e+004
4.4812e-013-2.6009e+004i     -6.1230e-017   2.6009e+004
3.1387e-013+7.5783e+003i     -6.1230e-017   7.5783e+003
3.1387e-013-7.5783e+003i     -6.1230e-017   7.5783e+003
2.4454e-013+3.7952e+002i     -7.2736e-016   3.7952e+002
2.4454e-013-3.7952e+002i     -7.2736e-016   3.7952e+002
       0                     -1.0000e+000   0
       0                     -1.0000e+000   0
-9.9504e-013+2.4715e+003i    3.8286e-016    2.4715e+003
-9.9504e-013-2.4715e+003i    3.8286e-016    2.4715e+003
-1.1766e-012+1.4892e+004i    1.6081e-016    1.4892e+004
-1.1766e-012-1.4892e+004i    1.6081e-016    1.4892e+004
```

Clearly, the spacecraft is *unstable* due to a pair of zero eigenvalues (we can ignore the negligible, positive real parts of some eigenvalues, and assume that those real parts are zeros). The natural frequencies of the spacecraft range from 0 to 47 588 rad/s. The nonzero natural frequencies denote structural vibration of the spacecraft. The control objective is to design a controller which stabilizes the spacecraft, and brings the transient response to zero within 5 s, with zero maximum overshoot, while requiring input torques not exceeding 0.1 N-m, when the spacecraft is initially perturbed by a hub rotation of 0.01 rad. due to the movement of astronauts. The initial condition corresponding to the initial perturbation caused by the astronauts' movement is $\mathbf{x}(0) = [0.01; \text{zeros}(1,25)]^T$. Let us see whether the spacecraft is controllable:

```
>>rank(ctrb(A,B)) <enter>

ans=

 6
```

Since the rank of the controllability test matrix is *less than* 26, the order of the plant, it follows that the spacecraft is *uncontrollable*. The *uncontrollable* modes are the structural vibration modes, while the *unstable* mode is the rigid-body rotation with zero natural frequency. Hence, the spacecraft is *stabilizable* and an optimal regulator can be designed for the spacecraft, since stabilizability of the plant is a sufficient condition for the existence of a unique, positive definite solution to the algebraic Riccati equation. Let us select $\mathbf{Q} = 200\mathbf{I}$, and $\mathbf{R} = \mathbf{I}$, noting that the size of $\mathbf{Q}$ is (26×26) while that of $\mathbf{R}$ is (3×3), and solve the Riccati equation using *lqr* as follows:

```
>>[Ko,Mo,E] = lqr(A,B,200*eye(26),eye(3)); <enter>
```

A positive definite solution to the algebraic Riccati equation *exists* for the present choice of $\mathbf{Q}$ and $\mathbf{R}$, *even though the plant is uncontrollable*. Due to the size of the plant, we avoid printing the solution, **Mo**, and the optimal feedback gain matrix, **Ko**, here, but the closed-loop eigenvalues, **E**, are the following:

```
E =
-1.7321e+003+4.7553e+004i
-1.7321e+003-4.7553e+004i
-1.7502e+003+4.7529e+004i
-1.7502e+003-4.7529e+004i
-1.8970e+003+2.5943e+004i
-1.8970e+003-2.5943e+004i
-1.8991e+003+2.5916e+004i
-1.8991e+003-2.5916e+004i
-1.8081e+003+1.4569e+004i
-1.8081e+003-1.4569e+004i
-1.8147e+003+1.4550e+004i
-1.8147e+003-1.4550e+004i
-7.3743e+002+7.6536e+003i
-7.3743e+002-7.6536e+003i
-7.3328e+002+7.6142e+003i
-7.3328e+002-7.6142e+003i
-2.6794e+002+2.5348e+003i
-2.6794e+002-2.5348e+003i
-2.5808e+002+2.4698e+003i
-2.5808e+002-2.4698e+003i
-3.9190e+001+3.7744e+002i
-3.9190e+001-3.7744e+002i
-1.1482e+000+4.3165e-001i
-1.1482e+000-4.3165e-001i
-1.8066e+001+2.0911e+002i
-1.8066e+001-2.0911e+002i
```

All the closed-loop eigenvalues (contained in the vector **E**) have negative real-parts, indicating that the closed-loop system is asymptotically stable, which is a bonus! Let us check whether the performance objectives are met by this design by calculating the closed-loop initial response as follows: