

```
>>sysCL=ss(A-B*Ko,zeros(26,3),C,D); <enter>
>>[y,t,X] = initial(sysCL, [0.01 zeros(1, 25)]'); u = -Ko*X'; <enter>
```

The closed-loop outputs, $y_1(t)$, $y_2(t)$, and $y_3(t)$, and the required torque inputs, $u_1(t)$, $u_2(t)$, and $u_3(t)$, are plotted in Figure 6.5. We see from Figure 6.5 that all the three outputs settle to zero in 5 s, with zero overshoot, and that the input torque magnitudes are smaller than 0.1 N-m, as desired. Therefore, our design is successful in meeting all the performance objectives and input effort limits.

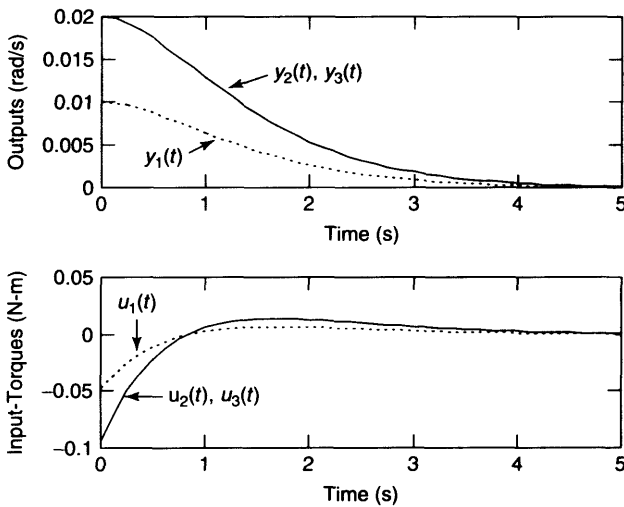


Figure 6.5 Closed-loop initial response and required inputs of the regulated flexible spacecraft for optimal regulator designed with $\mathbf{Q} = 200\mathbf{I}$ and $\mathbf{R} = \mathbf{I}$

Examples 6.1 and 6.2 illustrate two practical problems, one with a controllable plant, and the other with an uncontrollable plant, which are successfully solved using linear optimal control. Note the ease with which the two multi-input problems are solved when compared with the pole-placement approach (which, as you will recall, would have resulted in far too many design parameters than can be fixed by pole-placement). We will now consider the application of optimal control to more complicated problems.

6.3 Optimal Control of Tracking Systems

Consider a linear, time-varying plant with state-equation given by Eq. (6.1). It is required to design a tracking system for this plant if the desired state-vector, $\mathbf{x}_d(t)$, is the solution of the following equation:

$$\dot{\mathbf{x}}_d^{(1)}(t) = \mathbf{A}_d(t)\mathbf{x}_d(t) \quad (6.35)$$

Recall from Chapter 5 that the desired state dynamics is described by a homogeneous state-equation, because $\mathbf{x}_d(t)$ it is unaffected by the input, $\mathbf{u}(t)$. Subtracting Eq. (6.1) from Eq. (6.35), we get the following state-equation for the tracking-error, $\mathbf{e}(t) = \mathbf{x}_d(t) - \mathbf{x}(t)$:

$$\mathbf{e}^{(1)}(t) = \mathbf{A}(t)\mathbf{e}(t) + [\mathbf{A}_d(t) - \mathbf{A}(t)]\mathbf{x}_d(t) - \mathbf{B}(t)\mathbf{u}(t) \quad (6.36)$$

The control objective is to find the control input, $\mathbf{u}(t)$, such that the tracking-error, $\mathbf{e}(t)$, is brought to zero in the steady-state. To achieve this objective by optimal control, we have to first define the objective function to be minimized. Note that, as opposed to the regulator problem in which the input $\mathbf{u}(t) = -\mathbf{K}(t)\mathbf{x}(t)$, now the control input will also depend linearly on the desired state-vector, $\mathbf{x}_d(t)$. If we express Eqs. (6.1) and (6.35) by a *combined system* of which the state-vector is $\mathbf{x}_c(t) = [\mathbf{e}(t)^T; \mathbf{x}_d(t)^T]^T$, then the control input must be given by the following linear control-law:

$$\mathbf{u}(t) = -\mathbf{K}_c(t)\mathbf{x}_c(t) = -\mathbf{K}_c(t)[\mathbf{e}(t)^T; \mathbf{x}_d(t)^T]^T \quad (6.37)$$

where $\mathbf{K}_c(t)$ is the *combined feedback gain* matrix. Note that Eqs. (6.35) and (6.36) can be expressed as the following *combined state-equation*:

$$\mathbf{x}_c^{(1)}(t) = \mathbf{A}_c(t)\mathbf{x}_c(t) + \mathbf{B}_c(t)\mathbf{u}(t) \quad (6.38)$$

where

$$\mathbf{A}_c(t) = \begin{bmatrix} \mathbf{A}(t) & [\mathbf{A}_d(t) - \mathbf{A}(t)] \\ \mathbf{0} & \mathbf{A}_d(t) \end{bmatrix}; \quad \mathbf{B}_c(t) = \begin{bmatrix} -\mathbf{B}(t) \\ \mathbf{0} \end{bmatrix} \quad (6.39)$$

Since Eqs. (6.1) and (6.2) are now replaced by Eqs. (6.38) and (6.37), respectively, the objective function for the *combined system* can be expressed as an extension of Eq. (6.3) as follows:

$$J(t, t_f) = \int_t^{t_f} [\mathbf{x}_c^T(\tau)\mathbf{Q}_c(\tau)\mathbf{x}_c(\tau) + \mathbf{u}^T(\tau)\mathbf{R}(\tau)\mathbf{u}(\tau)] d\tau \quad (6.40)$$

Note that although we desire that the tracking error, $\mathbf{e}(t) = \mathbf{x}_d(t) - \mathbf{x}(t)$, be reduced to zero in the steady-state (i.e. when $t \rightarrow \infty$), we *cannot* pose the tracking system design as an optimal control problem with *infinite* control interval (i.e. $t_f = \infty$). The reason is that the desired state-vector, $\mathbf{x}_d(t)$ (hence $\mathbf{x}_c(t)$), *may not* go to zero in the steady-state, and thus a *non-zero* control input, $\mathbf{u}(t)$, may be required in the steady-state. Also, note that the combined system described by Eq. (6.38) is *uncontrollable*, because the desired state dynamics given by Eq. (6.35) is unaffected by the input, $\mathbf{u}(t)$. Therefore, the combined system's optimal control problem, represented by Eqs. (6.37)–(6.40) is *not guaranteed* to have a unique, positive definite solution. Hence, to have a guaranteed unique, positive definite solution to the optimal control problem, let us *exclude* the uncontrollable desired state-vector from the objective function, by choosing the *combined state-weighting matrix* as follows:

$$\mathbf{Q}_c(t) = \begin{bmatrix} \mathbf{Q}(t) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \quad (6.41)$$

which results in the following objective function:

$$J(t, t_f) = \int_t^{t_f} [\mathbf{e}^T(\tau)\mathbf{Q}(\tau)\mathbf{e}(\tau) + \mathbf{u}^T(\tau)\mathbf{R}(\tau)\mathbf{u}(\tau)] d\tau \quad (6.42)$$

which is the same as Eq. (6.3), with the crucial difference that $\mathbf{u}(t)$ in Eq. (6.42) is given by Eq. (6.37), rather than Eq. (6.2). By choosing $\mathbf{Q}(t)$ and $\mathbf{R}(t)$ to be positive semi-definite and positive definite, respectively, we satisfy the remaining sufficient conditions for the existence of a unique, positive definite solution to the optimal control problem. Note that the *optimal feedback gain matrix*, $\mathbf{K}_{oc}(t)$, is given by the following extension of Eq. (6.28):

$$\mathbf{K}_{oc}(t) = \mathbf{R}^{-1}(t)\mathbf{B}_c^T(t)\mathbf{M}_{oc} \quad (6.43)$$

where $\mathbf{M}_{oc}$ is the solution to the following Riccati equation:

$$-\partial\mathbf{M}_{oc}/\partial t = \mathbf{A}_c^T(t)\mathbf{M}_{oc} + \mathbf{M}_{oc}\mathbf{A}_c(t) - \mathbf{M}_{oc}\mathbf{B}_c(t)\mathbf{R}^{-1}(t)\mathbf{B}_c^T(t)\mathbf{M}_{oc} + \mathbf{Q}_c(t) \quad (6.44)$$

subject to the terminal condition, $\mathbf{M}_{oc}(t_f, t_f) = \mathbf{0}$. Since $\mathbf{M}_{oc}$ is symmetric (see Section 6.1), it can be expressed as

$$\mathbf{M}_{oc} = \begin{bmatrix} \mathbf{M}_{o1} & \mathbf{M}_{o2} \\ \mathbf{M}_{o2}^T & \mathbf{M}_{o3} \end{bmatrix} \quad (6.45)$$

where $\mathbf{M}_{o1}$ and $\mathbf{M}_{o2}$ correspond to the plant and the desired state dynamics, respectively. Substituting Eqs. (6.45) and (6.39) into Eq. (6.43), we can express the optimal feedback gain matrix as follows:

$$\mathbf{K}_{oc}(t) = -[\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{M}_{o1}; \quad \mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{M}_{o2}] \quad (6.46)$$

and the optimal control input is thus obtained by substituting Eq. (6.46) into Eq. (6.37) as follows:

$$\mathbf{u}(t) = \mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{M}_{o1}\mathbf{e}(t) + \mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{M}_{o2}\mathbf{x}_d(t) \quad (6.47)$$

Note that Eq. (6.47) does not require the sub-matrix, $\mathbf{M}_{o3}$. The individual matrix differential equations to be solved for $\mathbf{M}_{o1}$ and $\mathbf{M}_{o2}$ can be obtained by substituting Eqs. (6.39) and (6.45) into Eq. (6.44) as follows:

$$-\partial\mathbf{M}_{o1}/\partial t = \mathbf{A}^T(t)\mathbf{M}_{o1} + \mathbf{M}_{o1}\mathbf{A}(t) - \mathbf{M}_{o1}\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{M}_{o1} + \mathbf{Q}(t) \quad (6.48)$$

$$\begin{aligned} -\partial\mathbf{M}_{o2}/\partial t &= \mathbf{M}_{o2}\mathbf{A}_d(t) + \mathbf{M}_{o1}[\mathbf{A}_d(t) - \mathbf{A}(t)] \\ &+ [\mathbf{A}^T(t) - \mathbf{M}_{o1}\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)]\mathbf{M}_{o2} \end{aligned} \quad (6.49)$$

Note that Eq. (6.48) is identical to the *matrix Riccati equation*, Eq. (6.29), which can be solved *independently* of Eq. (6.49), without taking into account the desired state dynamics. Once the optimal matrix, $\mathbf{M}_{o1}$, is obtained from the solution of Eq. (6.48), it can be substituted into Eq. (6.49), which can then be solved for $\mathbf{M}_{o2}$. Equation (6.49) is a *linear, matrix differential equation*, and can be written as follows:

$$-\partial\mathbf{M}_{o2}/\partial t = \mathbf{M}_{o2}\mathbf{A}_d(t) + \mathbf{M}_{o1}[\mathbf{A}_d(t) - \mathbf{A}(t)] + \mathbf{A}_{CL}^T(t)\mathbf{M}_{o2} \quad (6.50)$$

where $\mathbf{A}_{CL}(t) = \mathbf{A}(t) - \mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{M}_{01}$, the closed-loop state-dynamics matrix. The solution of the optimal tracking system thus requires the solution of the linear differential equation, Eq. (6.50), in addition to the solution of the optimal regulator problem given by Eq. (6.48). The solutions of Eqs. (6.48) and (6.49) are subject to the terminal condition, $\mathbf{M}_{0c}(t_f, t_f) = \mathbf{0}$, which results in $\mathbf{M}_{01}(t_f, t_f) = \mathbf{0}$, and $\mathbf{M}_{02}(t_f, t_f) = \mathbf{0}$.

Often, it is required to track a *constant* desired state-vector, $\mathbf{x}_d(t) = \mathbf{x}_d^c$, which implies $\mathbf{A}_d(t) = \mathbf{0}$. Then the matrices $\mathbf{M}_{01}$ and $\mathbf{M}_{02}$ are both constants in the *steady-state* (i.e. $t_f \rightarrow \infty$), and are the solutions of the following *steady-state* equations (obtained by setting $\partial \mathbf{M}_{01} / \partial t = \partial \mathbf{M}_{02} / \partial t = \mathbf{0}$ and $\mathbf{A}_d(t) = \mathbf{0}$ in Eqs. (6.48) and (6.49)):

$$\mathbf{0} = \mathbf{A}^T(t)\mathbf{M}_{01} + \mathbf{M}_{01}\mathbf{A}(t) - \mathbf{M}_{01}\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{M}_{01} + \mathbf{Q}(t) \quad (6.51)$$

$$\mathbf{0} = -\mathbf{M}_{01}\mathbf{A}(t) + \mathbf{A}_{CL}^T(t)\mathbf{M}_{02} \quad (6.52)$$

We immediately recognize Eq. (6.51) as the *algebraic Riccati equation* (Eq. (6.32)) of the *steady-state, optimal regulator* problem. From Eq. (6.52), we must have

$$\mathbf{M}_{02} = [\mathbf{A}_{CL}^T(t)]^{-1}\mathbf{M}_{01}\mathbf{A}(t) \quad (6.53)$$

where $\mathbf{M}_{01}$ is the solution to the algebraic Riccati equation, Eq. (6.51). Note, however, that even though $\mathbf{M}_{01}$ and $\mathbf{M}_{02}$ are finite constants in the steady-state, the matrix $\mathbf{M}_{0c}$ is *not* a finite constant in the steady-state, because as $\mathbf{x}(t_f)$ tends to a constant desired state in the limit $t_f \rightarrow \infty$, the objective function (Eq. (6.42)) becomes *infinite*, hence a steady-state solution to Eq. (6.44) does not exist. The only way $\mathbf{M}_{0c}$ can *not* be a finite constant (when both $\mathbf{M}_{01}$ and $\mathbf{M}_{02}$ are finite constants) is when $\mathbf{M}_{03}$ (the discarded matrix in Eq. (6.45)) is *not* a finite constant in the steady-state.

Substituting Eq. (6.53) into Eq. (6.47), we get the following input for the *constant* desired state vector, $\mathbf{x}_d^c$:

$$\mathbf{u}(t) = \mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{M}_{01}\mathbf{e}(t) + \mathbf{R}^{-1}(t)\mathbf{B}^T(t)[\mathbf{A}_{CL}^T(t)]^{-1}\mathbf{M}_{01}\mathbf{A}(t)\mathbf{x}_d^c \quad (6.54)$$

Substituting Eq. (6.54) into Eq. (6.36), we get the following closed-loop tracking error state-equation with $\mathbf{A}_d(t) = \mathbf{0}$ and $\mathbf{x}_d(t) = \mathbf{x}_d^c$:

$$\mathbf{e}^{(1)}(t) = \mathbf{A}_{CL}(t)\mathbf{e}(t) - [\mathbf{A}(t) + \mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\{\mathbf{A}_{CL}^T(t)\}^{-1}\mathbf{M}_{01}\mathbf{A}(t)]\mathbf{x}_d^c \quad (6.55)$$

From Eq. (6.55), it is clear that the tracking error can go to *zero* in the steady-state (i.e. as $t \rightarrow \infty$) for *any non-zero, constant* desired state, $\mathbf{x}_d^c$, if $\mathbf{A}_{CL}(t)$ is *asymptotically stable* and

$$\mathbf{A}(t) + \mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\{\mathbf{A}_{CL}^T(t)\}^{-1}\mathbf{M}_{01}\mathbf{A}(t) = \mathbf{0} \quad (6.56)$$

or

$$\mathbf{M}_{01}\mathbf{A}(t) = -[\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\{\mathbf{A}_{CL}^T(t)\}^{-1}]^{-1}\mathbf{A}(t) = -\mathbf{A}_{CL}^T(t)\{\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\}^{-1}\mathbf{A}(t) \quad (6.57)$$

Equation (6.57) can be expanded to give the following equation to be satisfied by $\mathbf{M}_{o1}$ for the steady-state tracking error to be zero:

$$\mathbf{M}_{o1}\mathbf{A}(t) = -\mathbf{A}^T(t)\{\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\}^{-1}\mathbf{A}(t) + \mathbf{M}_{o1}\mathbf{A}(t) \quad (6.58)$$

which implies that $\mathbf{A}^T(t)\{\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\}^{-1}\mathbf{A}(t) = \mathbf{0}$, or $\{\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\}^{-1} = \mathbf{0}$. Clearly, this is an *impossible* requirement, because it implies that $\mathbf{R}(t) = \mathbf{0}$. Hence, we *cannot* have an optimal tracking system in which the tracking error, $\mathbf{e}(t)$, goes to zero in the steady-state for *any* constant desired state-vector, $\mathbf{x}_d^c$. As in Chapter 5, the best we can do is to have $\mathbf{e}(t)$ going to zero for *some* values of $\mathbf{x}_d^c$. However, if we want this to happen *while satisfying* the optimality condition for $\mathbf{M}_{o2}$ given by Eq. (6.53), we will be left with the requirement that $[\mathbf{A}(t) + \mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\{\mathbf{A}_{CL}^T(t)\}^{-1}\mathbf{M}_{o1}\mathbf{A}(t)]\mathbf{x}_d^c = \mathbf{0}$ for some non-zero $\mathbf{x}_d^c$, resulting in $\{\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\}^{-1}\mathbf{A}(t)\mathbf{x}_d^c = \mathbf{0}$, which implies that $\{\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\}^{-1}$ must be a *singular matrix* – again, an impossible requirement. Therefore, the only possible way we can ensure that the tracking error goes to zero for some desired state is by *dropping the optimality condition* on $\mathbf{M}_{o2}$ given by Eq. (6.53). Then we can write the input vector as follows:

$$\mathbf{u}(t) = \mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{M}_{o1}\mathbf{e}(t) - \mathbf{K}_d(t)\mathbf{x}_d^c \quad (6.59)$$

where $\mathbf{K}_d(t)$ is the (non-optimal) feedforward gain matrix which would make $\mathbf{e}(t)$ zero in the steady-state for some values of $\mathbf{x}_d^c$. Substituting Eq. (6.59) into Eq. (6.36) we get the following state-equation for the tracking error:

$$\mathbf{e}^{(1)}(t) = \mathbf{A}_{CL}(t)\mathbf{e}(t) - [\mathbf{A}(t) - \mathbf{B}(t)\mathbf{K}_d(t)]\mathbf{x}_d^c \quad (6.60)$$

Equation (6.60) implies that for a zero steady-state tracking error, $\mathbf{K}_d(t)$ must be selected such that $[\mathbf{A}(t) - \mathbf{B}(t)\mathbf{K}_d(t)]\mathbf{x}_d^c = \mathbf{0}$. The closed-loop state-dynamics matrix, $\mathbf{A}_{CL}$, in Eq. (6.60) is an optimal matrix given by $\mathbf{A}_{CL}(t) = \mathbf{A}(t) - \mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{M}_{o1}$, where $\mathbf{M}_{o1}$ is the solution to the algebraic Riccati equation, Eq. (6.51). Hence, the design of a tracking system *does not* end with finding a unique, positive definite solution, $\mathbf{M}_{o1}$, to the algebraic Riccati equation (which would make $\mathbf{A}_{CL}(t)$ asymptotically stable); we should also find a (non-optimal) feedforward gain matrix, $\mathbf{K}_d(t)$, such that $[\mathbf{A}(t) - \mathbf{B}(t)\mathbf{K}_d(t)]\mathbf{x}_d^c = \mathbf{0}$ for some values of the constant desired state-vector, $\mathbf{x}_d^c$. Note that if the plant has *as many inputs* as there are state variables, then $\mathbf{B}(t)$ is a square matrix, and it would be possible to make $\mathbf{e}(t)$ zero in the steady-state for *any arbitrary* $\mathbf{x}_d^c$, by choosing $\mathbf{K}_d(t) = \mathbf{B}^{-1}(t)\mathbf{A}(t)$ (provided $\mathbf{B}(t)$ is non-singular, i.e. the plant is *controllable*.)

Example 6.3

Consider the amplifier-motor of Example 3.7, with the numerical values given as $J = 1 \text{ kg.m}^2$, $R = 1000 \text{ ohms}$, $L = 100 \text{ henry}$, $a = 0.3 \text{ kg.m}^2/\text{s}^2/\text{Ampere}$, and $K_A = 10$. Recall from Example 3.7 that the state-vector of the amplifier-motor is $\mathbf{x}(t) = [\theta(t); \theta^{(1)}(t); i(t)]^T$, where $\theta(t)$ is the angular position of the load on the motor, and $i(t)$ is the current supplied to the motor. The input vector is $\mathbf{u}(t) = [v(t); T_L(t)]^T$, where $v(t)$ is the input voltage to the amplifier and $T_L(t)$ is the torque applied by the load on the motor. It is desired to design a tracking system

such that the load on the motor moves from an initial angular position, $\theta(0) = 0$, to desired angular position $\theta_d(t) = 0.1$ rad. in about six seconds, and comes to rest at the desired position. The maximum angular velocity of the load, $\theta^{(1)}(t)$, should not exceed 0.05 rad/s. After the load comes to rest at the desired position, the current supplied to the motor should be zero. The desired state-vector is thus $\mathbf{x}_d^c = [0.1; 0; 0]^T$. The plant's state coefficient matrices are the following:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -0.01 & 0.3 \\ 0 & -0.003 & -10 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \\ 0.1 & 0 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}; \quad \mathbf{D} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (6.61)$$

The eigenvalues of the linear, time-invariant plant are calculated as follows:

```
>>damp(A) <enter>
```

Eigenvalue	Damping	Freq. (rad/sec)
0	-1.0000	0
-0.0101	1.0000	0.0101
-9.9999	1.0000	9.9999

The plant is stable, with an eigenvalue at the origin. Since the plant is time-invariant, the controller gain matrices must be constants. Let us first find the optimal feedback gain matrix, $\mathbf{K}_{o1} = -\mathbf{R}^{-1}\mathbf{B}^T\mathbf{M}_{o1}$ by choosing $\mathbf{Q} = \mathbf{I}$ and $\mathbf{R} = \mathbf{I}$, and solving the algebraic Riccati equation as follows:

```
>>[Ko1,Mo1,E] = lqr(A,B,eye(3),eye(2)) <enter>
```

```
Ko1 =
    0.0025    0.0046    0.0051
   -1.0000   -1.7220   -0.0462
```

```
Mo1 =
    1.7321    1.0000    0.0254
    1.0000    1.7220    0.0462
    0.0254    0.0462    0.0513
```

```
E =
   -10.0004
  -0.8660 + 0.5000i
  -0.8660 - 0.5000i
```

The closed-loop eigenvalues are all in the left-half plane, as desired for asymptotic stability of the tracking error dynamics. Next, we calculate the feedforward gain matrix, $\mathbf{K}_d$, which will make the steady-state tracking error zero for the specified constant desired state, $\mathbf{x}_d^c$. This is done by selecting $\mathbf{K}_d$ such that $\mathbf{A}\mathbf{x}_d^c = \mathbf{B}\mathbf{K}_d\mathbf{x}_d^c$ as follows:

$$\mathbf{K}_d = \begin{bmatrix} K_{d1} & K_{d2} & K_{d3} \\ K_{d4} & K_{d5} & K_{d6} \end{bmatrix} \quad (6.62)$$

$$\mathbf{BK}_d \mathbf{x}_d^c = \begin{bmatrix} 0 \\ -0.1 K_{d4} \\ 0.01 K_{d1} \end{bmatrix} \quad (6.63)$$

Therefore, $\mathbf{Ax}_d^c = \mathbf{BK}_d \mathbf{x}_d^c$ implies that $K_{d1} = 0$ and $K_{d4} = 0$. We can also choose the remaining elements of $\mathbf{K}_d$ as zeros, and still satisfy $\mathbf{Ax}_d^c = \mathbf{BK}_d \mathbf{x}_d^c$. Hence, $\mathbf{K}_d = \mathbf{0}$, and the control input is given by $\mathbf{u}(t) = \mathbf{K}_0 \mathbf{e}(t)$.

Let us obtain the tracking error response of the system to the initial tracking error, $\mathbf{e}(0) = \mathbf{x}_d^c - \mathbf{x}(0) = \mathbf{x}_d^c$ (since $\mathbf{x}(0) = \mathbf{0}$) as follows:

```
>>t=0:0.05:6; sysCL=ss(A-B*Ko1, zeros(3,2),C,D); [y,t,e]
= initial(sysCL, [0.1 0 0]'); <enter>
```

Then, the state-vector, $\mathbf{x}(t)$, of the closed-loop system can be calculated using $\mathbf{x}(t) = \mathbf{x}_d^c - \mathbf{e}(t)$ as follows:

```
>>n=size(t, 1); for i=1:n; Xd(i,:)= [0.1 0 0]; end; X=Xd-e; <enter>
```

while the input vector, $\mathbf{u}(t)$, is calculated as

```
>>u = Ko1*e'; <enter>
```

The calculated state variables, $\theta(t)$, $\theta^{(1)}(t)$, $i(t)$, the input voltage, $v(t)$, and the loading torque, $T_L(t)$, are plotted in Figure 6.6. Note that all the state variables reach their desired values in about 6 s, with a maximum overshoot in angular

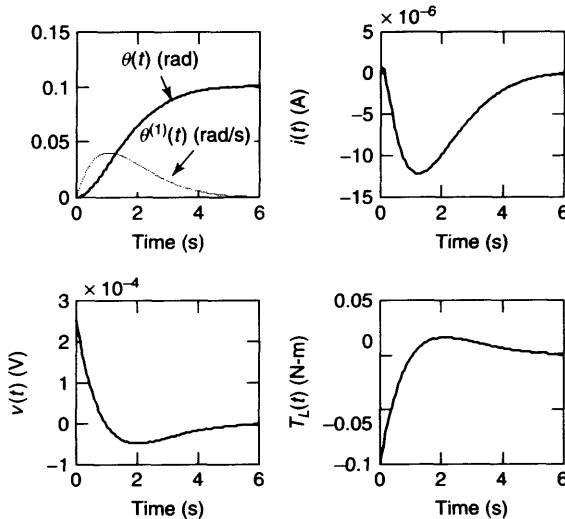


Figure 6.6 Closed-loop initial response of the tracking system for amplifier-motor, with constant desired state, $\theta(t) = 0.1$ rad, $\theta^{(1)}(t) = 0$ rad, and $i(t) = 0$ amperes

velocity, $\theta^{(1)}(t)$, of about 0.04 rad/s, and the maximum overshoot in current, $i(t)$, of -12.5×10^{-6} A. The maximum input voltage, $v(t)$, is 2.5×10^{-4} V and the maximum loading torque is -0.1 N-m.

Using SIMULINK, let us now investigate the effect of an uncertainty in the amplifier gain, K_A , on the tracking system. A SIMULINK block-diagram of the closed-loop tracking system is shown in Figure 6.7. The uncertainty in the amplifier gain, ΔK_A , affects only the third state-variable, $i(t)$, and is incorporated into the

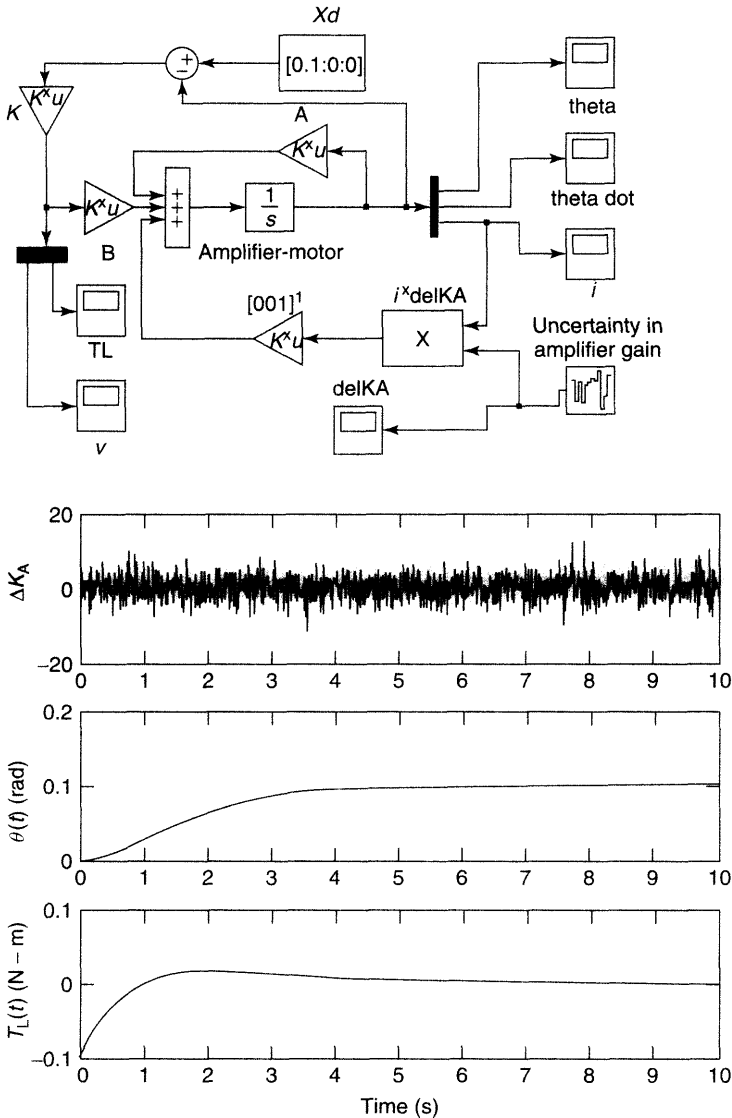


Figure 6.7 Simulation of the tracking system for amplifier-motor, with uncertainty in the amplifier gain, K_A

plant dynamics model using the *band-limited white noise* block output, ΔK_A , multiplied with the vector $[0; 0; i(t)]^T$, and added to the summing junction, which results in the plant dynamics being represented as $\mathbf{x}^{(1)}(t) = (\mathbf{A} + \Delta\mathbf{A})\mathbf{x}(t) + \mathbf{B}u(t)$, where

$$\Delta\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Delta K_A \end{bmatrix} \quad (6.64)$$

The simulated values of ΔK_A , $\theta(t)$, and $T_L(t)$ are shown in Figure 6.7. Note that despite a random variation in K_A between ± 10 , the tracking system's performance is unaffected. This signifies a design which is quite robust to variations in K_A .

Example 6.4

For a particular set of flight conditions, the lateral dynamics of an aircraft are described by a linear, time-invariant state-space representation with the following coefficient matrices:

$$\mathbf{A} = \begin{bmatrix} -9.75 & 0 & -9.75 & 0 \\ 0 & -0.8 & 8 & 0 \\ 0 & -1 & -0.8 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 20 & 2.77 \\ 0 & -3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (6.65)$$

The state-vector consists of the *roll-rate*, $p(t)$, *yaw-rate*, $r(t)$, *side-slip angle*, $\beta(t)$, and *bank angle*, $\phi(t)$, and is written as $\mathbf{x}(t) = [p(t); r(t); \beta(t); \phi(t)]^T$. The input vector consists of the *aileron deflection angle*, $\delta_A(t)$, and *rudder deflection angle*, $\delta_R(t)$, i.e. $\mathbf{u}(t) = [\delta_A(t); \delta_R(t)]^T$. It is desired to execute a *steady turn* with a constant yaw-rate, $r_d(t) = 0.05$ rad/s, a constant bank angle, $\phi_d(t) = 0.02$ rad, and zero roll-rate and sideslip angle, $p_d(t) = \beta_d(t) = 0$. The desired state-vector is thus $\mathbf{x}_d^c = [0; 0.05; 0; 0.02]^T$. The desired state must be reached in about two seconds, with a maximum roll-rate, $p(t)$, less than 0.1 rad/s and the control inputs ($\delta_A(t)$ and $\delta_R(t)$) not exceeding 0.3 rad. Let us first select a feedforward gain matrix, $\mathbf{K}_d$, which satisfies $\mathbf{A}\mathbf{x}_d^c = \mathbf{B}\mathbf{K}_d\mathbf{x}_d^c$ as follows:

$$\mathbf{K}_d = \begin{bmatrix} K_{d1} & K_{d2} & K_{d3} & K_{d4} \\ K_{d5} & K_{d6} & K_{d7} & K_{d8} \end{bmatrix} \quad (6.66)$$

$$\mathbf{B}\mathbf{K}_d\mathbf{x}_d^c = \begin{bmatrix} 20(0.05K_{d2} + 0.02K_{d4}) + 2.77(0.05K_{d6} + 0.02K_{d8}) \\ -3(0.05K_{d6} + 0.1K_{d8}) \\ 0 \\ 0 \end{bmatrix}; \quad \mathbf{A}\mathbf{x}_d^c = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (6.67)$$

Equation (6.67) indicates that, to make $\mathbf{A}\mathbf{x}_d^c = \mathbf{B}\mathbf{K}_d\mathbf{x}_d^c$, we must have $0.05K_{d6} + 0.02K_{d8} = 0$ and $0.05K_{d2} + 0.02K_{d4} = 0$, which can be satisfied by selecting $\mathbf{K}_d = \mathbf{0}$. It now remains for us to calculate the optimal feedback gain matrix, $\mathbf{K}_{o1}$, by solving the algebraic Riccati equation. Note that the plant is stable with the following eigenvalues:

```
>>damp(A) <enter>
```

Eigenvalue	Damping	Freq. (rad/sec)
0	-1.0000	0
-0.8000 + 2.8284i	0.2722	2.9394
-0.8000 - 2.8284i	0.2722	2.9394
-9.7500	1.0000	9.7500

Let us select $\mathbf{R} = \mathbf{I}$. After experimenting with several values of $\mathbf{Q}$, we select the following which satisfies the desired transient response and input limitations:

$$\mathbf{Q} = \begin{bmatrix} 20 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 200 \end{bmatrix} \quad (6.68)$$

With this combination of $\mathbf{Q}$ and $\mathbf{R}$, the algebraic Riccati equation is solved as follows:

```
>>[Ko1,Mo1,E] = lqr(A,B,Q,R) <enter>
Ko1 =
  4.1351    0.3322   -0.2935   14.0420
  0.5229   -2.7651   -1.1575    1.6797

Mo1 =
  0.2068    0.0166   -0.0147    0.7021
  0.0166    0.9370    0.3723    0.0884
 -0.0147    0.3723    3.6355   -0.0908
  0.7021    0.0884   -0.0908   65.7890

E =
 -90.7740
  -8.4890
  -3.1456
  -1.3872
```

The initial response of the tracking error, $\mathbf{e}(t)$, to $\mathbf{x}(0) = \mathbf{0}$ is calculated as follows:

```
>>sysCL=ss(A-B*Ko1,zeros(4,2),eye(4),zeros(4,2)); <enter>
>>t=0:0.001:2; [y,t,e]=initial(sysCL,[0 0.05 0 0.02]',t); <enter>
```

and the state-vector and control input vector are then calculated as

```
>>n = size(t,1); for i=1:n; Xd(i,:) = [0 0.05 0 0.02]; end; X = Xd-e;
  u = -Ko1*e'; <enter>
```

The calculated state-vector, $\mathbf{x}(t)$, and input vector, $\mathbf{u}(t)$, are plotted in Figures 6.8 and 6.9, respectively. Note that all the transients settle to their desired values in about two seconds, with a maximum overshoot in roll-rate, $p(t)$, of less than 0.06 rad/s, and a small maximum side-slip angle, $\beta(t)$. The aileron and rudder deflections are seen in Figure 6.9 to be less than 0.3 rad. as desired.