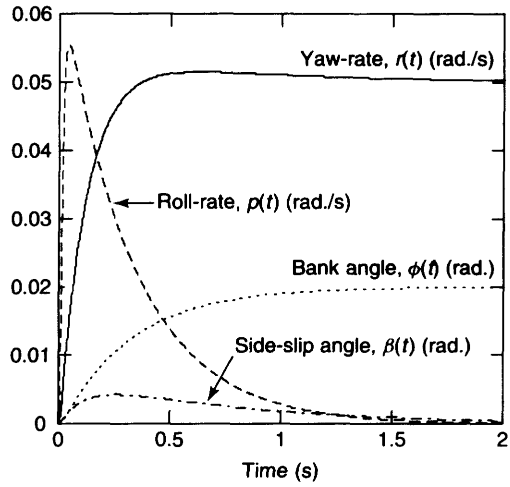

Figure 6.8 Response of the tracking system for aircraft lateral dynamics for a desired steady turn with turn-rate, $r(t) = 0.05$ rad. and bank angle, $\phi(t) = 0.02$ rad

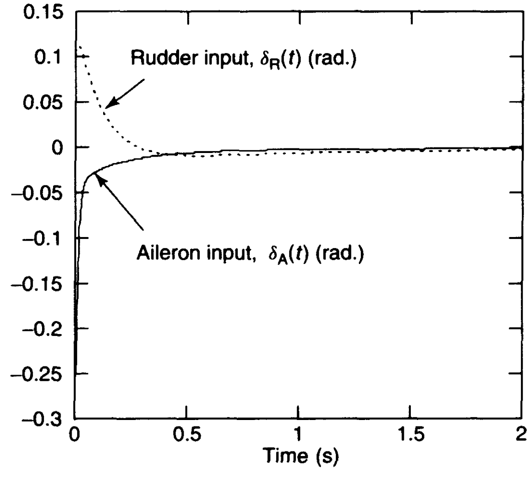

Figure 6.9 Aileron and rudder inputs of the aircraft lateral dynamics tracking system for achieving a desired steady turn with turn-rate, $r(t) = 0.05$ rad. and bank angle, $\phi(t) = 0.02$ rad

6.4 Output Weighted Linear Optimal Control

Many times it is the output, $\mathbf{y}(t)$, rather than the state-vector, $\mathbf{x}(t)$, which is included in the objective function for minimization. The reason for this may be either a lack of physical understanding of some state variables, which makes it difficult to assign weightage to them, or that the desired performance objectives are better specified in terms of the measured output; remember that it is the *output* of the system (rather than the *state*) which indicates the performance to an observer (either a person or a mathematical device discussed in Chapter 5). When the output is used to define the performance, the objective function can be expressed as follows:

$$J(t, t_f) = \int_t^{t_f} [\mathbf{y}^{\mathrm{T}}(\tau)\mathbf{Q}(\tau)\mathbf{y}(\tau) + \mathbf{u}^{\mathrm{T}}(\tau)\mathbf{R}(\tau)\mathbf{u}(\tau)]\, d\tau \tag{6.69}$$

where $\mathbf{Q}(t)$ is now the *output weighting matrix*. Substituting the output equation given by

$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t) \tag{6.70}$$

into Eq. (6.69), the objective function becomes the following:

$$\begin{aligned} J(t, t_f) = \int_t^{t_f} & [\mathbf{x}^{\mathrm{T}}(\tau)\mathbf{C}^{\mathrm{T}}(\tau)\mathbf{Q}(\tau)\mathbf{C}(\tau)\mathbf{x}(\tau) + \mathbf{x}^{\mathrm{T}}(\tau)\mathbf{C}^{\mathrm{T}}(\tau)\mathbf{Q}(\tau)\mathbf{D}(\tau)\mathbf{u}(\tau) \\ & + \mathbf{u}^{\mathrm{T}}(\tau)\mathbf{D}^{\mathrm{T}}(\tau)\mathbf{Q}(\tau)\mathbf{C}(\tau)\mathbf{x}(\tau) + \mathbf{u}^{\mathrm{T}}(\tau)\{\mathbf{R}(\tau) + \mathbf{D}^{\mathrm{T}}(\tau)\mathbf{Q}(\tau)\mathbf{C}(\tau)\}\mathbf{u}(\tau)]\, d\tau \end{aligned} \tag{6.71}$$

or

$$\begin{aligned} J(t, t_f) = \int_t^{t_f} & [\mathbf{x}^{\mathrm{T}}(\tau)\mathbf{Q_G}(\tau)\mathbf{x}(\tau) + \mathbf{x}^{\mathrm{T}}(\tau)\mathbf{S}(\tau)\mathbf{u}(\tau) + \mathbf{u}^{\mathrm{T}}(\tau)\mathbf{S}^{\mathrm{T}}(\tau)\mathbf{x}(\tau) \\ & + \mathbf{u}^{\mathrm{T}}(\tau)\mathbf{R_G}(\tau)\mathbf{u}(\tau)]\, d\tau \end{aligned} \tag{6.72}$$

where $\mathbf{Q_G}(\tau) = \mathbf{C}^{\mathrm{T}}(\tau)\mathbf{Q}(\tau)\mathbf{C}(\tau)$, $\mathbf{S}(\tau) = \mathbf{C}^{\mathrm{T}}(\tau)\mathbf{Q}(\tau)\mathbf{D}(\tau)$, and $\mathbf{R_G}(\tau) = \mathbf{R}(\tau) + \mathbf{D}^{\mathrm{T}}(\tau)\mathbf{Q}(\tau)\mathbf{C}(\tau)$. You can show, using steps similar to Sections 6.1 and 6.2, that the optimal regulator gain matrix, $\mathbf{K_o}(t)$, which minimizes $J(t, t_f)$ given by Eq. (6.72) can be expressed as

$$\begin{aligned} \mathbf{K_o}(t) = \mathbf{R}_{\mathbf{G}}^{-1}(t)[\mathbf{B}^{\mathrm{T}}(t)\mathbf{M_o} + \mathbf{S}^{\mathrm{T}}(t)] = \{\mathbf{R}(t) + \mathbf{D}^{\mathrm{T}}(t)\mathbf{Q}(t)\mathbf{C}(t)\}^{-1}[\mathbf{B}^{\mathrm{T}}(t)\mathbf{M_o} \\ + \mathbf{D}^{\mathrm{T}}(\tau)\mathbf{Q}(\tau)\mathbf{C}(\tau)] \end{aligned} \tag{6.73}$$

where $\mathbf{M_o}(t, t_f)$ is the solution to the following matrix Riccati equation:

$$\begin{aligned} -\partial\mathbf{M_o}/\partial t = \mathbf{A}_{\mathbf{G}}^{\mathrm{T}}(t)\mathbf{M_o} + \mathbf{M_o}\mathbf{A_G}(t) - \mathbf{M_o}\mathbf{B}(t)\mathbf{R}_{\mathbf{G}}^{-1}(t)\mathbf{B}^{\mathrm{T}}(t)\mathbf{M_o} \\ + [\mathbf{Q_G}(t) - \mathbf{S}(t)\mathbf{R_G}^{-1}(t)\mathbf{S}^{\mathrm{T}}(t)] \end{aligned} \tag{6.74}$$

where $\mathbf{A_G}(t) = \mathbf{A}(t) - \mathbf{B}(t)\mathbf{R}_{\mathbf{G}}^{-1}(t)\mathbf{S}^{\mathrm{T}}(t)$. Equation (6.74) can be solved numerically in a manner similar to that for the solution to Eq. (6.29), with the terminal condition $\mathbf{M_o}(t_f, t_f) = \mathbf{0}$.

The steady-state optimal control problem (i.e. when $t_f \to \infty$) results in the following algebraic Riccati equation:

$$\mathbf{0} = \mathbf{A}_{\mathbf{G}}^{\mathrm{T}}(t)\mathbf{M_o} + \mathbf{M_o}\mathbf{A_G}(t) - \mathbf{M_o}\mathbf{B}(t)\mathbf{R}_{\mathbf{G}}^{-1}(t)\mathbf{B}^{\mathrm{T}}(t)\mathbf{M_o} + [\mathbf{Q_G}(t) - \mathbf{S}(t)\mathbf{R}_{\mathbf{G}}^{-1}(t)\mathbf{S}^{\mathrm{T}}(t)] \tag{6.75}$$

The sufficient conditions for the existence of a unique, positive definite solution, $\mathbf{M_o}$, to Eq. (6.75) are similar to those for Eq. (6.32), i.e. the system – whose state coefficient matrices are $\mathbf{A_G}(t)$ and $\mathbf{B}(t)$ – is *controllable*, $[\mathbf{Q_G}(t) - \mathbf{S}(t)\mathbf{R}_{\mathbf{G}}^{-1}(t)\mathbf{S}^{\mathrm{T}}(t)]$ is a *positive* *semi-definite* matrix, and $\mathbf{R_G}(t)$ is a *positive definite* matrix. However, note that for a plant with state-equation given by Eq. (6.1), these conditions are *more restrictive* than those for the existence of a unique, positive definite solution to Eq. (6.32).

Solution to the algebraic Riccati equation, Eq. (6.75), can be obtained using the MATLAB function *are* or CST function *lqr*, by appropriately specifying the coefficient matrices $\mathbf{A_G}(t)$ and $\mathbf{B}(t)$, and the weighting matrices $[\mathbf{Q_G}(t) - \mathbf{S}(t)\mathbf{R_G^{-1}}(t)\mathbf{S^T}(t)]$ and $\mathbf{R_G}(t)$ at each instant of time, t. However, MATLAB (CST) provides the function *lqry* for solving the output weighted linear, quadratic optimal control problem, which only needs the plant coefficient matrices $\mathbf{A}(t)$, $\mathbf{B}(t)$, $\mathbf{C}(t)$, and $\mathbf{D}(t)$, and the output and control weighting matrices, $\mathbf{Q}(t)$ and $\mathbf{R}(t)$, respectively, as follows:

```
>>[K,Mo,E] = lqry(sys,Q,R) <enter>
```

where *sys* is the state-space LTI object of the plant. The *lqry* command is thus easier to use than either *are* or *lqr* for solving the output weighted problem.

Example 6.5

Let us design an optimal regulator for the flexible bomber aircraft (Examples 4.7, 6.1) using output weighting. Recall that the two outputs of this sixth-order, time-invariant plant are the normal acceleration in m/s^2, $y_1(t)$, and the pitch-rate in rad./s, $y_2(t)$. Our performance objectives remain the same as in Example 6.1, i.e. a maximum overshoot of less than ± 2 m/s^2 in the normal-acceleration and less than ± 0.03 rad/s in pitch-rate, and a settling time less than 5 s, while requiring elevator and canard deflections (the two inputs) *not exceeding* ± 0.1 rad. (5.73°), if the initial condition is 0.1 rad/s perturbation in the pitch-rate ($\mathbf{x}(0) = [0; 0.1; 0; 0; 0; 0]^T$). After some trial and error, we find that the following weighting matrices satisfy the performance requirements:

$$\mathbf{Q}(t) = \begin{bmatrix} 0.0001 & 0 \\ 0 & 1 \end{bmatrix}; \quad \mathbf{R}(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{6.76}$$

which result in the following optimal gain matrix, $\mathbf{K_o}(t)$, the solution to algebraic Riccati equation, $\mathbf{M_o}$, and the closed-loop eigenvalue vector, $\mathbf{E}$:

```
>>sys=ss(A,B,C,D);[Ko,Mo,E] = lqry(sys,[0.0001 0; 0 1],eye(2)) <enter>

Ko=
 1.2525e+001 -2.3615e-002 3.3557e-001 -3.7672e-003 1.0426e+001 4.4768e+000
 6.6601e+000 3.3449e-001 1.8238e-001 -1.1227e-002 5.9690e+000 2.6042e+000

Mo=
 1.1557e+000  4.6221e-001 2.1004e-002  7.2281e-003  1.6700e-001  6.6601e-002
 4.6221e-001  1.1715e+000 1.3859e-002  1.2980e-002 -3.1487e-004  3.3449e-003
 2.1004e-002  1.3859e-002 7.3455e-004  1.9823e-004  4.4742e-003  1.8238e-003
 7.2281e-003  1.2980e-002 1.9823e-004  1.9469e-004 -5.0229e-005 -1.1227e-004
 1.6700e-001 -3.1487e-004 4.4742e-003 -5.0229e-005  1.3902e-001  5.9690e-002
 6.6601e-002  3.3449e-003 1.8238e-003 -1.1227e-004  5.9690e-002  2.6042e-002
```

```
E =
-1.1200e+003
-9.4197e+001
-1.5385e+000+ 2.2006e+000i
-1.5385e+000- 2.2006e+000i
-1.0037e+000+ 6.2579e-001i
-1.0037e+000- 6.2579e-001i
```

The closed-loop initial response and required control inputs are calculated as follows:

```
>>sysCL=ss(A-B*Ko,zeros(6,2),C,D); [y, t, X]=initial(sysCL, [0 0.1 0 0 0 0]');
  u = -Ko*X'; <enter>
```

Figure 6.10 shows the plots of the calculated outputs and inputs. Note that all the performance objectives are met, and the maximum control values are −0.035 rad. for both desired elevator and canard deflections, $u_1(t)$ and $u_2(t)$, respectively.

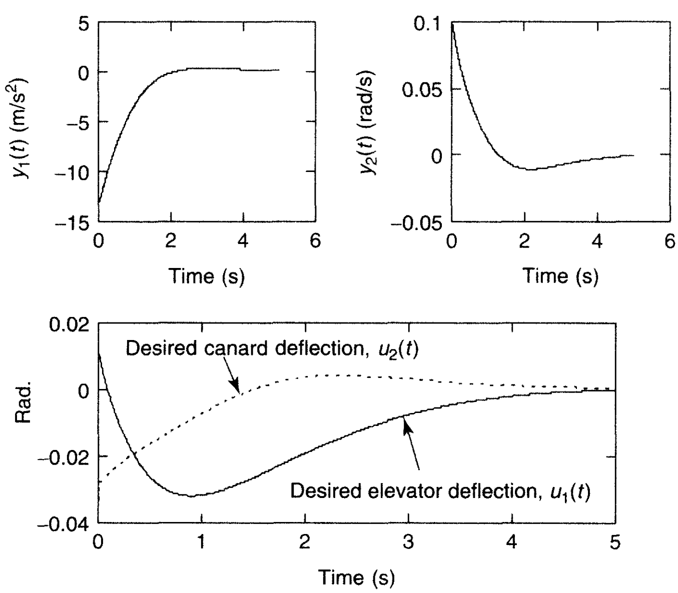

Figure 6.10 Closed-loop initial response and required inputs for the flexible bomber aircraft, with optimal regulator designed using output weighting

Let us simulate the closed-loop initial response of the present design with the regulator designed in Example 6.1 using state-weighting. We use the same SIMULINK block-diagram as shown in Figure 6.3, with the measurement noise modeled by the *band-limited white noise* block of power 10^{-4}. The resulting simulated response is shown in Figure 6.11. Note that the fluctuations in both $y_1(t)$ and $y_2(t)$ are about an *order of magnitude* smaller than those observed in Figure 6.3, which indicates that a better *robustness* with respect to measurement noise has been achieved using output-weighted optimal control.