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Kalman Filters

7.1 Stochastic Systems

We defined *deterministic systems* in Chapter 1 as those systems whose governing physical laws are such that if the state of the system at some time (i.e. the *initial condition*) is specified, then one can precisely predict the state at a later time. Most (if not all) natural processes are *non-deterministic* systems (i.e. systems that are *not* deterministic). *Non-deterministic* systems can be divided into two categories: *stochastic* and *random* systems. A *stochastic* (also called *probabilistic*) system has such governing physical laws that even if the initial conditions are known precisely, it is impossible to determine the system's state at a later time. In other words, based upon the *stochastic* governing laws and the initial conditions, one could only determine the *probability* of a state, rather than the state itself. When we flip a perfect coin, we do not know if head or tail will come up; we only know that both the possibilities have an equal *probability* of 50%. The disturbances encountered by many physical systems – such as atmospheric turbulence and disturbance due to an uneven ground – are produced by *stochastic* systems. A *random* system is one which has no apparent governing physical laws. While it is a human endeavour to ascribe physical laws to observed natural phenomena, some natural phenomena are so complex that it is impossible to pin down the physical laws obeyed by them. The human brain presently appears to be a *random* system. Environmental temperature and rainfall are outputs of a random system. It is very difficult to practically distinguish between *random* and *stochastic* systems. Also, frequently we are unable to practically distinguish between a non-deterministic (stochastic or random) system, and a deterministic system whose future state we *cannot* predict based upon an *erroneous measurement* of the initial condition. A *double pendulum* (Figure 1.5) is a classic example of *unpredictable*, deterministic systems. For all practical purposes, we will treat all unpredictable systems – deterministic or non-deterministic – (stochastic or random) as *stochastic* systems, since we have to employ the same statistical methods while studying such systems, regardless of the nature of their physical governing laws. For the same reason, it is a common practice to use the words *random*, *stochastic*, and *unpredictable* interchangeably. While we are unable to predict the state of a random process, we can evolve a strategy to deal with such processes when they affect a control system in the form of noise. Such a strategy has to be based on a branch of mathematics dealing with unpredictable systems, called *statistics*.

Since the initial state, $\mathbf{x}(t_0)$, of a stochastic system is insufficient to determine the future state, $\mathbf{x}(t)$, we have to make an *educated guess* as to what the future state might be, based upon a statistical analysis of many similar systems, and taking the *average* of

their future states at a given time, t . For example, if we want to know how a stray dog would behave if offered a bone, we will offer N stray dogs N different bones, and record the state variables of interest, such as intensity of the sound produced by each dog, the forward acceleration, the angular position of the dog's tail, and, perhaps, the intensity of the dog's bite, as functions of time. Suppose $\mathbf{x}_i(t)$ is the recorded state-vector of the i th dog. Then the *mean* state-vector is defined as follows:

$$\mathbf{x}_m(t) = (1/N) \sum_{i=1}^N \mathbf{x}_i(t) \quad (7.1)$$

Note that $\mathbf{x}_m(t)$ is the state-vector we would *expect* after studying N similar stochastic systems. Hence, it is also called the *expected value* of the state-vector, $\mathbf{x}(t)$, and denoted by $\mathbf{x}_m(t) = E[\mathbf{x}(t)]$. The *expected value operator*, $E[\cdot]$, has the following properties (which are clear from Eq. (7.1)):

- (a) $E[\text{random signal}] = \text{mean of the random signal}$.
- (b) $E[\text{deterministic signal}] = \text{deterministic signal}$.
- (c) $E[\mathbf{x}_1(t) + \mathbf{x}_2(t)] = E[\mathbf{x}_1(t)] + E[\mathbf{x}_2(t)]$.
- (d) $E[\mathbf{C}\mathbf{x}(t)] = \mathbf{C}E[\mathbf{x}(t)]$; $\mathbf{C} = \text{constant matrix}$.
- (e) $E[\mathbf{x}(t)\mathbf{C}] = E[\mathbf{x}(t)]\mathbf{C}$; $\mathbf{C} = \text{constant matrix}$.
- (f) $E[\mathbf{x}(t)\mathbf{y}(t)] = E[\mathbf{x}(t)]\mathbf{y}(t)$; $\mathbf{x}(t) = \text{random signal}$; $\mathbf{y}(t) = \text{deterministic signal}$.
- (g) $E[\mathbf{y}(t)\mathbf{x}(t)] = \mathbf{y}(t)E[\mathbf{x}(t)]$; $\mathbf{x}(t) = \text{random signal}$; $\mathbf{y}(t) = \text{deterministic signal}$.

We can define another statistical quantity, namely a *correlation matrix* of the state-vector as follows:

$$\mathbf{R}_x(t, \tau) = (1/N) \sum_{i=1}^N \mathbf{x}_i(t)\mathbf{x}_i^T(\tau) \quad (7.2)$$

The correlation matrix, $\mathbf{R}_x(t, \tau)$, is a measure of a statistical property called *correlation* among the different state variables, as well as between the same state variable at two different times. For two scalar variables, $x_1(t)$ and $x_2(t)$, if the expected value of $x_1(t)x_2(\tau)$ is zero, i.e. $E[x_1(t)x_2(\tau)] = 0$, where τ is different from t , then $x_1(t)$ and $x_2(t)$ are said to be *uncorrelated*. Comparing Eqs. (7.1) and (7.2), it is clear that the correlation matrix is the expected value of the matrix $\mathbf{x}_i(t)\mathbf{x}_i^T(\tau)$, or $\mathbf{R}_x(t, \tau) = E[\mathbf{x}_i(t)\mathbf{x}_i^T(\tau)]$. When $t = \tau$, the correlation matrix, $\mathbf{R}_x(t, t) = E[\mathbf{x}_i(t)\mathbf{x}_i^T(t)]$, is called the *covariance matrix*. It is obvious that the covariance matrix, $\mathbf{R}_x(t, t)$, is symmetric. If $\mathbf{R}_x(t, \tau)$ is a *diagonal* matrix, it implies that all the state variables are *uncorrelated*, i.e. $E[x_i(t)x_j(\tau)] = 0$, where $i \neq j$. You are referred to a textbook on probability and statistics, such as that by Papoulis [1], for further details on expected values, correlation and covariance matrices.

There are special stochastic systems, called *stationary systems*, for which all the statistical properties, such as the mean value, $\mathbf{x}_m(t)$, and correlation matrix, $\mathbf{R}_x(t, \tau)$, do not

change with a *translation* in time, i.e. when time, t , is replaced by $(t + \theta)$. Hence, for a *stationary system*, $\mathbf{x}_m(t + \theta) = \mathbf{x}_m(t) = \text{constant}$, and $\mathbf{R}_x(t + \theta, \tau + \theta) = \mathbf{R}_x(t, \tau)$ for *all* values of θ . Expressing $\mathbf{R}_x(t, \tau) = \mathbf{R}_x(t, t + \alpha)$ where $\tau = t + \alpha$, we can show that for a stationary system, $\mathbf{R}_x(t - \alpha, t) = \mathbf{R}_x(t, t + \alpha)$, which implies that for a stationary system, the correlation matrix is only a function of the *time-shift*, α , i.e. $\mathbf{R}_x(t, t + \alpha) = \mathbf{R}_x(\alpha)$. Many stochastic systems of interest are assumed to be stationary, which greatly simplifies the statistical analysis of such systems.

The expected value, $\mathbf{x}_m(t)$, and the correlation matrix, $\mathbf{R}_x(t, \tau)$, are examples of *ensemble statistical properties*, i.e. properties of an *ensemble* (or *group*) of N *samples*. Clearly, the accuracy by which the expected value, $\mathbf{x}_m(t)$, approximates the actual state-vector, $\mathbf{x}(t)$, depends upon the number of samples, N . If N is increased, the accuracy is improved. For a *random system*, an *infinite* number of samples are required for predicting the state-vector, i.e. $N = \infty$. However, we can usually obtain good accuracy with a *finite* (but large) number of samples. Of course, the samples must be taken in as many different situations as possible. For example, if we confined our sample of stray dogs to our own neighborhood, the accuracy of ensemble properties would suffer. Instead, we should pick the dogs from many different parts of the town, and repeat our experiment at various times of the day, month, and year. However, as illustrated by the stray dog example, one has to go to great lengths merely to collect sufficient data for arriving at an accurate ensemble average. Finding an ensemble average in some cases may even be impossible, such as trying to calculate the expected value of annual rainfall in London – which would require constructing N Londons and taking the ensemble average of the annual rainfall recorded in all the Londons! However, we can measure annual rainfall in London for *many years*, and take the *time average* by dividing the total rainfall by the number of years. Taking a time average is entirely different from taking the ensemble average, especially if the system is *non-stationary*. However, there is a sub-class of stationary systems, called *ergodic systems*, for which a *time average* is the same as an ensemble average. For those stationary systems that are *not* ergodic, it is inaccurate to substitute a time average for the ensemble average, but we still do so routinely because there is no other alternative (there is only one London in the world). Hence, we will substitute time averaged statistics for ensemble statistics of *all* stationary systems. For a stationary system, by taking the time average, we can evaluate the *mean*, $\mathbf{x}_m$, and the *correlation matrix*, $\mathbf{R}_x(\tau)$, over a *large* time period, $T \rightarrow \infty$, as follows:

$$\mathbf{x}_m = \lim_{T \rightarrow \infty} (1/T) \int_{-T/2}^{T/2} \mathbf{x}(t) dt \quad (7.3)$$

$$\mathbf{R}_x(\tau) = \lim_{T \rightarrow \infty} (1/T) \int_{-T/2}^{T/2} \mathbf{x}(t) \mathbf{x}^T(t + \tau) dt \quad (7.4)$$

Note that since the system is stationary, the mean value, $\mathbf{x}_m$, is a constant, and the *correlation matrix*, $\mathbf{R}_x(\tau)$, is only a function of the *time-shift*, τ . For frequency domain analysis of stochastic systems, it is useful to define a *power spectral density matrix*, $\mathbf{S}_x(\omega)$, as the Fourier transform of the correlation matrix, $\mathbf{R}_x(\tau)$, given by

$$\mathbf{S}_x(\omega) = \int_{-\infty}^{\infty} \mathbf{R}_x(\tau) e^{-i\omega\tau} d\tau \quad (7.5)$$

where ω is the frequency of excitation (i.e. the frequency of an oscillatory input applied to the stochastic system). The power spectral density matrix, $\mathbf{S}_x(\omega)$, is a measure of how the *power* of a random signal, $\mathbf{x}(t)$, varies with frequency, ω . The Fourier transform of the random signal, $\mathbf{x}(t)$, is given by

$$X(i\omega) = \int_{-\infty}^{\infty} \mathbf{x}(t)e^{-i\omega t} dt \quad (7.6)$$

It can be shown from Eqs. (7.4)–(7.6) that

$$\mathbf{S}_x(\omega) = X(i\omega)X^T(-i\omega) \quad (7.7)$$

The correlation matrix, $\mathbf{R}_x(\tau)$, can be obtained by calculating the inverse Fourier transform of the power spectral density matrix, $\mathbf{S}_x(\omega)$, as follows:

$$\mathbf{R}_x(\tau) = (1/2\pi) \int_{-\infty}^{\infty} \mathbf{S}_x(\omega)e^{i\omega\tau} d\omega \quad (7.8)$$

Comparing Eqs. (7.4) and (7.8), we find that at $\tau = 0$, the correlation matrix becomes the *covariance matrix*, given by

$$\mathbf{R}_x(0) = \lim_{T \rightarrow \infty} (1/T) \int_{-T/2}^{T/2} \mathbf{x}(t)\mathbf{x}^T(t) dt = (1/2\pi) \int_{-\infty}^{\infty} \mathbf{S}_x(\omega)e^{i\omega\tau} d\omega \quad (7.9)$$

or

$$\int_{-\infty}^{\infty} \mathbf{S}_x(\omega)e^{i\omega\tau} d\omega = 2\pi \mathbf{x}_{ms} \quad (7.10)$$

where $\mathbf{x}_{ms}$, called the *mean-square value* of $\mathbf{x}(t)$, is the following:

$$\mathbf{x}_{ms} = \lim_{T \rightarrow \infty} (1/T) \int_{-T/2}^{T/2} \mathbf{x}(t)\mathbf{x}^T(t) dt \quad (7.11)$$

Usually, the state-vector, $\mathbf{x}(t)$, is available at *discrete* time points, rather than as a *continuous* function of time. The *discrete* Fourier transform of a *discrete* time state-vector, $\mathbf{x}(j\Delta t)$, where Δt is the time step size and $j = 1, 2, \dots, N$, is calculated by the following expression:

$$X(k\Delta\omega) = \sum_{j=1}^N \mathbf{x}(j\Delta t)e^{-2\pi(j-1)(k-1)/N} \quad (7.12)$$

where $\Delta\omega = 2\pi/(N\Delta t)$ is the frequency step size. Similarly, the *inverse* discrete Fourier transform can be calculated as follows:

$$\mathbf{x}(j\Delta t) = (1/N) \sum_{k=1}^N X(k\Delta\omega)e^{2\pi(j-1)(k-1)/N} \quad (7.13)$$

When the discrete Fourier transform is used to calculate the *power spectral density*, the result must be divided by N , the number of frequency points in $X(k\Delta\omega)$ as follows:

$$\mathbf{S}_x(k\Delta\omega) = X(k\Delta\omega)X^T(-k\Delta\omega)/N \quad (7.14)$$

The discrete Fourier transform of a signal, $\mathbf{x}(t)$, can be calculated by using the MATLAB command for discrete Fourier transform, *fft*, as follows:

```
>>X = fft(x,n) <enter>
```

where $\mathbf{x}$ is the returned n -point *discrete* Fourier transform of $\mathbf{x}$. If n is not specified, an N -point Fourier transform is calculated, where N is the number of discrete time points in the matrix $\mathbf{x}$. $\mathbf{x}$ has as many rows as there are time points, and as many columns as there are variables. Thus, each column of $\mathbf{x}$ is the Fourier transform of the corresponding column of $\mathbf{x}$. The inverse discrete Fourier transform is similarly calculated using the MATLAB command *ifft* as follows:

```
>>x = ifft(X,n) <enter>
```

Example 7.1

Consider a scalar random signal, $x(t)$, which is generated using the MATLAB command *randn* as follows:

```
>>t=0:0.1:10; randn('seed',0); x=randn(size(t)); <enter>
```

The command *randn* generates a random number, according to a special random process with a *zero* mean value, called *normal(or Gaussian) probability distribution* [1]. The 'seed' of the random number generator, *randn*, is set to zero to initialize the generating process to the value when MATLAB is started. The random signal, $x(t)$, is generated in time steps of 0.1 seconds, for 10 seconds – a total of 101 time points. The full-order discrete Fourier transform of $x(t)$ is calculated as follows:

```
>>X=fft(x); <enter>
```

The discrete frequencies, ω , at which the Fourier transform of $x(t)$ is calculated are calculated and stored in vector w as follows:

```
>>w = (0: length(X)-1)'*2*pi/(0.1*length(X)); <enter>
```

The power spectral density, $S_x(\omega)$, of the discrete random signal, $x(t)$, is calculated as follows:

```
>>S = X.*conj(X)/length(X); <enter>
```

The correlation function, $R_x(\tau)$ is calculated by taking the inverse Fourier transform of $S(\omega)$ with the help of the MATLAB command *ifft* as follows:

```
>>Rx = ifft(S); <enter>
```

The scalar plots $x(t)$, $|X(\omega)|$, $S_x(\omega)$, and $R_X(t)$ are shown in Figure 7.1. Note the special shape of the $S_x(\omega)$ plot – it is symmetrical about the *mid-point frequency* $\omega = 51\Delta\omega = 31.1$ rad/s. This is a characteristic of the normal distribution.

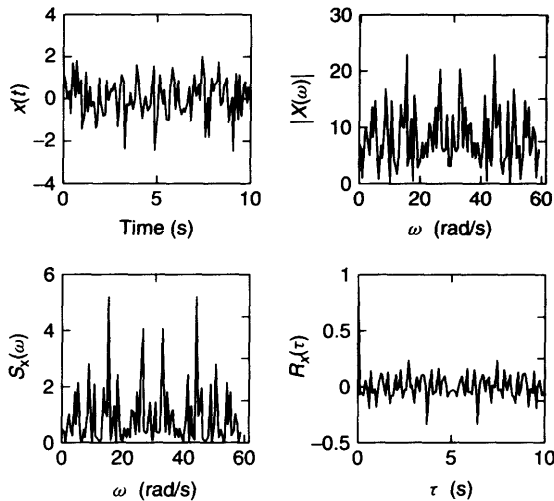


Figure 7.1 Random signal, $x(t)$, its Fourier transform magnitude, $|X(\omega)|$, power spectral density, $S_x(\omega)$, and correlation function, $R_x(\tau)$

The mean value of $x(t)$ can be obtained using the MATLAB command *mean* as follows:

```
>>mean(x) <enter>
ans =
    0.0708
```

which is very close to zero as expected (the mean value of $x(t)$ will be *exactly* zero only if the number of sample points, i.e. the length of $x(t)$, is *infinity*). The *covariance*, $R_x(0)$ can be read from the plot of $R_x(\tau)$ to be approximately 0.9, or obtained more precisely using the MATLAB command *cov* as follows:

```
>>cov(x) <enter>
ans =
    0.9006
```

The *mean-square* of $x(t)$, x_{ms} , can be calculated as follows:

```
>>xms = mean(x.*x) <enter>
ans =
    0.8967
```

Note that x_{ms} is equal to the *mean value* of the power spectral density $S_x(\omega)$ plot, calculated as follows:

```
>>mean(S) <enter>
ans =
    0.8967
```