7.3 White Noise, and White Noise Filters

In the previous section, we saw how random signals can be generated using a random number generator, such as the MATLAB command *randn*, which generates random numbers with a *normal (or Gaussian) probability distribution*. If we could theoretically use a random number generator to generate an *infinite set of random numbers* with *zero mean value* and a *normal probability distribution*, we would have a *stationary* random signal with a *constant* power spectral density [1]. Such a theoretical random signal with zero mean and a constant power spectral density is called a *stationary white noise*. A *white noise vector*, $\mathbf{w}(t)$, has each element as a white noise, and can be regarded as the state-vector of a stochastic system called a *white noise process*. For such a process, we can write the mean value, $\mathbf{w_m} = 0$, and the power spectral density matrix, $\mathbf{S}(\omega) = \mathbf{W}$, where $\mathbf{W}$ is a *constant* matrix. Since the power spectral density of a white noise is a constant, it follows from Eq. (7.9) that the *covariance matrix* of the white noise is a matrix with all elements as *infinities*, i.e. $\mathbf{R_w}(0) = \infty$. Also, from Eq. (7.8) we can write the following expression for the *correlation matrix* of the white noise process (recall that the *inverse* Laplace (or Fourier) transform of a constant is the constant multiplied by the *unit impulse function*, $\delta(t)$):

$$\mathbf{R_w}(\tau) = \mathbf{W}\delta(\tau) \tag{7.18}$$

Equation (7.18) re-iterates that the covariance of white noise is infinite, i.e. $\mathbf{R_w}(0) = \infty$. Note that the correlation matrix of white noise is *zero*, i.e. $\mathbf{R_w}(\tau) = \mathbf{0}$, for $\tau \neq 0$, which implies that the white noise is *uncorrelated in time* (there is absolutely no correlation between $\mathbf{w}(t)$ and $\mathbf{w}(t+\tau)$). Hence, white noise can be regarded as *perfectly* random. However, a physical process with constant power spectral density is unknown; all physical processes have a power spectrum that tends to zero as $\omega \to \infty$. All known physical processes have a finite *bandwidth*, i.e. range of frequencies at which the process can be *excited* (denoted by peaks in the power spectrum). The white noise, in contrast, has an *infinite* bandwidth. Hence, the white noise process appears to be a figment of our imagination. However, it is a useful figment of imagination, as we have seen in Chapters 5 and 6 that white noise can be used to approximate random disturbances while simulating the response of control systems.

Let us see what happens if a linear, time-invariant system is placed in the path of a white noise. Such a linear system, into which the white noise is input, would be called a *white noise filter*. Since the input is a white noise, from the previous section we expect that the output of the filter would be a random signal. What are the statistical properties of such a random signal? We can write the following expression for the output, $\mathbf{y}(t)$, of a linear system with transfer matrix, $\mathbf{G}(s)$, with a white noise input, $\mathbf{w}(t)$, using the matrix form of the *superposition integral* of Eq. (2.120), as follows:

$$\mathbf{y}(t) = \int_{-\infty}^{t} \mathbf{g}(t-\tau)\mathbf{w}(\tau)\,d\tau \tag{7.19}$$

where $\mathbf{g}(t)$ is the *impulse response matrix* of the filter related to the transfer matrix, $\mathbf{G}(s)$, by the inverse Laplace transform as follows:

$$\mathbf{g}(t) = \int_0^{\infty} \mathbf{G}(s)e^{st}\,ds \tag{7.20}$$

The lower limit of integration in Eq. (7.19) can be changed to zero if the white noise, $\mathbf{w}(t)$, starts acting on the system at $t = 0$, and the system is *causal*, i.e. it starts producing the output *only after* receiving the input, but *not before that* (most physical systems are causal). Since the input white noise is a stationary process (with mean, $\mathbf{w_m} = 0$), and the filter is time-invariant, we expect that the output, $\mathbf{y}(t)$, should be the output of a stationary process, and we can determine its mean value by taking time average as follows:

$$\begin{aligned}
\mathbf{y_m} &= \lim_{T\to\infty}(1/T)\int_{-T/2}^{T/2}\mathbf{y}(t)\,dt = \lim_{T\to\infty}(1/T)\int_{-T/2}^{T/2}\left[\int_{-\infty}^{t}\mathbf{g}(t-\tau)\mathbf{w}(\tau)\,d\tau\right]dt \\
&= \int_{-\infty}^{t}\mathbf{g}(t-\tau)\left[\lim_{T\to\infty}(1/T)\int_{-T/2}^{T/2}\mathbf{w}(t)\,dt\right]d\tau = \int_{-\infty}^{t}\mathbf{g}(t-\tau)\mathbf{w_m}d\tau = 0
\end{aligned} \tag{7.21}$$

Hence, the mean of the output of a linear system for a white noise input is zero. The correlation matrix of the output signal, $\mathbf{R_y}(\tau)$ can be calculated as follows, using the properties of the expected value operator, $E[\cdot]$:

$$\begin{aligned}
\mathbf{R_y}(\tau) &= E[\mathbf{y}(t)\mathbf{y}^{\mathbf{T}}(t+\tau)] = E\left[\int_{-\infty}^{t}\mathbf{g}(t-\alpha)\mathbf{w}(\alpha)\,d\alpha\cdot\int_{-\infty}^{t+\tau}\mathbf{w}^{\mathbf{T}}(\beta)\mathbf{g}^{\mathbf{T}}(t+\tau-\beta)\,d\beta\right] \\
&= E\left[\int_{-\infty}^{t}\int_{-\infty}^{t+\tau}\mathbf{g}(t-\alpha)\mathbf{w}(\alpha)\mathbf{w}^{\mathbf{T}}(\beta)\mathbf{g}^{\mathbf{T}}(t+\tau-\beta)\,d\alpha\,d\beta\right] \\
&= \int_{-\infty}^{t}\int_{-\infty}^{t+\tau}\mathbf{g}(t-\alpha)E[\mathbf{w}(\alpha)\mathbf{w}^{\mathbf{T}}(\beta)]\mathbf{g}^{\mathbf{T}}(t+\tau-\beta)\,d\alpha\,d\beta
\end{aligned} \tag{7.22}$$

Using the fact that $E[\mathbf{w}(\alpha)\mathbf{w}^{\mathbf{T}}(\beta)] = \mathbf{R_w}(\alpha-\beta) = \mathbf{W}\delta(\beta-\alpha)$, we can write

$$\begin{aligned}
\mathbf{R_y}(\tau) &= \int_{-\infty}^{t}\int_{-\infty}^{t+\tau}\mathbf{g}(t-\alpha)\mathbf{W}\delta(\beta-\alpha)\mathbf{g}^{\mathbf{T}}(t+\tau-\beta),\,d\alpha\,d\beta \\
&= \int_{-\infty}^{t}\mathbf{g}(t-\alpha)\mathbf{W}\mathbf{g}^{\mathbf{T}}(t+\tau-\alpha)\,d\alpha = \int_{-\infty}^{t+\tau}\mathbf{g}(\lambda)\mathbf{W}\mathbf{g}^{\mathbf{T}}(\lambda+\tau)\,d\lambda
\end{aligned} \tag{7.23}$$

Since $\mathbf{y}(t)$ is the output of a stationary process, we can replace the upper limit of integration in Eq. (7.23) by ∞, provided $\mathbf{y}(t)$ reaches a *steady-state* as $t\to\infty$, i.e. the filter is *asymptotically stable*. Therefore, the correlation matrix of the output of an asymptotically stable system to the white noise can be written as follows:

$$\mathbf{R_y}(\tau) = \int_{-\infty}^{\infty}\mathbf{g}(\lambda)\mathbf{W}\mathbf{g}^{\mathbf{T}}(\lambda+\tau)\,d\lambda \tag{7.24}$$

The power spectral density matrix, $\mathbf{S_y}(\omega)$, for the output, $\mathbf{y}(t)$, can then be obtained by taking the Fourier transform of the correlation matrix, $\mathbf{R_y}(\tau)$, as follows:

$$\mathbf{S_y}(\omega) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \mathbf{g}(\lambda)\mathbf{W}\mathbf{g}^{\mathbf{T}}(\lambda + \tau)\, d\lambda \right] \mathrm{e}^{-i\omega\tau} d\tau \tag{7.25}$$

Inverting the order of integration in Eq. (7.25), we can write

$$\mathbf{S_y}(\omega) = \int_{-\infty}^{\infty} \mathbf{g}(\lambda)\mathbf{W} \left[\int_{-\infty}^{\infty} \mathbf{g}^{\mathbf{T}}(\lambda + \tau)\mathrm{e}^{-i\omega\tau} d\tau \right] d\lambda \tag{7.26}$$

The inner integral in Eq. (7.26) can be expressed as follows:

$$\int_{-\infty}^{\infty} \mathbf{g}^{\mathbf{T}}(\lambda + \tau)\mathrm{e}^{-i\omega\tau} d\tau = \int_{-\infty}^{\infty} \mathbf{g}^{\mathbf{T}}(\xi)\mathrm{e}^{-i\omega(\xi-\lambda)} d\xi = \mathrm{e}^{i\omega\lambda}\mathbf{G}^{\mathbf{T}}(i\omega) \tag{7.27}$$

where $\mathbf{G}(i\omega)$ is the *frequency response matrix* of the filter, i.e. $\mathbf{G}(i\omega) = \mathbf{G}(s = i\omega)$. Substituting Eq. (7.27) into Eq. (7.26), we get the following expression for the power spectral density matrix:

$$\mathbf{S_y}(\omega) = \left[\int_{-\infty}^{\infty} \mathbf{g}(\lambda)\mathrm{e}^{i\omega\lambda} d\lambda \right] \mathbf{W}\mathbf{G}^{\mathbf{T}}(i\omega) = \mathbf{G}(-i\omega)\mathbf{W}\mathbf{G}^{\mathbf{T}}(i\omega) \tag{7.28}$$

Equation (7.28) is an important result for the output of the white noise filter, and shows that the filtered white noise *does not* have a constant power spectral density, but one which depends upon the frequency response, $\mathbf{G}(i\omega)$, of the filter.

Example 7.4

Let us determine the power spectral density of white noise after passing through the first order low-pass filter of Eq. (7.15). The frequency response of this single-input, single-output filter is the following:

$$G(i\omega) = \omega_0/(i\omega + \omega_0) \tag{7.29}$$

Using Eq. (7.28), we can write the power spectral density as follows:

$$S(\omega) = [\omega_0/(-i\omega + \omega_0)]W[\omega_0/(i\omega + \omega_0)] = W\omega_0^2/(\omega^2 + \omega_0^2) \tag{7.30}$$

Equation (7.30) indicates that the spectrum of the white noise passing through this filter is no longer flat, but begins to decay at $\omega = \omega_0$. For this reason, the filtered white noise is called *colored noise* whose 'color' is indicated by the frequency, ω_0. The correlation function is obtained by taking the inverse Fourier transform of Eq. (7.30) as follows:

$$\begin{aligned} R_y(\tau) &= (1/2\pi) \int [W\omega_0^2/(\omega^2 + \omega_0^2)]\mathrm{e}^{i\omega\tau} d\omega \\ &= (W\omega_0/2)\exp(-\omega_0|\tau|) \quad (\tau > 0) \end{aligned} \tag{7.31}$$

Since for a stationary process, $R_y(-\tau) = R_y(\tau)$, we can write a general expression for $R_y(\tau)$ valid for all values of τ as follows:

$$R_y(\tau) = (W\omega_0/2)\exp(-\omega_0|\tau|) \tag{7.32}$$

Finally, the covariance matrix of the filtered white noise, $R_y(0)$, is given by

$$R_y(0) = (W\omega_0/2) \tag{7.33}$$

Example 7.5

Atmospheric turbulence is a random process, which has been studied extensively. A semi-empirical model of *vertical gust velocity*, $x(t)$, caused by atmospheric turbulence is the *Dryden spectrum* with the following expression for the power spectral density:

$$S_x(\omega) = a^2T[1 + 3(\omega T)^2]/[1 + (\omega T)^2]^2 \tag{7.34}$$

where a and T are constants. The correlation function, $R_x(\tau)$, of the Dryden spectrum can be calculated by taking the inverse Fourier transform of $S_x(\omega)$ according to Eq. (7.8), either analytically or using MATLAB. It can be shown that the analytical expression of $R_x(\tau)$ is the following:

$$R_x(\tau) = a^2(1 - 0.5|\tau|/T)\mathrm{e}^{-|\tau|/T} \tag{7.35}$$

It is customary to express stochastic systems as filters of white noise. What is the *transfer function* of a filter through which white noise must be passed to get the filtered output as the Dryden turbulence? We can employ Eq. (7.28) for the relationship between the power spectral density, $S_x(\omega)$, and the filter transfer function, $G(s)$, which can be written for the scalar case as follows:

$$S_x(\omega) = G(-i\omega)WG(i\omega) = WG(-i\omega)G(i\omega) \tag{7.36}$$

Comparing Eqs. (7.34) and (7.36), we see that a *factorization* of the power spectral density, $S_x(\omega)$, given by

$$\begin{aligned} S_x(\omega) &= a^2T[1 - \sqrt{3}T(i\omega)][1 + \sqrt{3}T(i\omega)]/\{[1 - T(i\omega)]^2[1 + T(i\omega)]^2\} \\ &= WG(-i\omega)G(i\omega) \end{aligned} \tag{7.37}$$

leads to the following possibilities for $G(s)$:

$$G(s) = a(T/W)^{1/2}(1 \pm \sqrt{3}Ts)/(1 \pm Ts)^2 \tag{7.38}$$

Since we want a stable filter (i.e. all poles of $G(s)$ in the left-half plane), we should have a '+' sign in the denominator of $G(s)$ in Eq. (7.38). Also, we do not want a *zero* in the right-half plane, because it leads to an undesirable phase plot of $G(i\omega)$, called *non-minimum phase*. A stable transfer function with all zeros in the left-half plane is called *minimum phase*. Hence, the following choice of $G(s)$ would yield a stable and minimum phase filter:

$$G(s) = a(T/W)^{1/2}(1+\sqrt{3}Ts)/(1+Ts)^2 \tag{7.39}$$

Such a method of obtaining the filter's transfer function through a factorization of the power spectral density is called *spectral factorization*, and is commonly employed in deriving white noise filter representations of stochastic systems. For $W = 1$ and $a = 1$, we can obtain a state-space representation of the filter with $T = 1$ s using MATLAB as follows:

```
>> T=1; n = sqrt(T)*[T*sqrt(3) 1]; d = conv([T 1],[T 1]); sysfilt = tf
  (n,d) <enter>

Transfer function:
 1.732 s+1
--------
s^2+2 s+1
```

A simulated response of the filter to white noise, representing the vertical gust velocity, $x(t)$, due to Dryden turbulence, can be obtained using MATLAB as follows:

```
>>t=0:0.001:1; u = randn(size(t)); [x,t,X] = lsim(sysfilt,u,t);
  <enter>
```

Figure 7.6 shows the Dryden turbulence power spectrum, $S(\omega)/a^2$ plotted against ωT, the Bode plots of the white noise filter transfer function, $G(s)$, for $T = 0.1$,

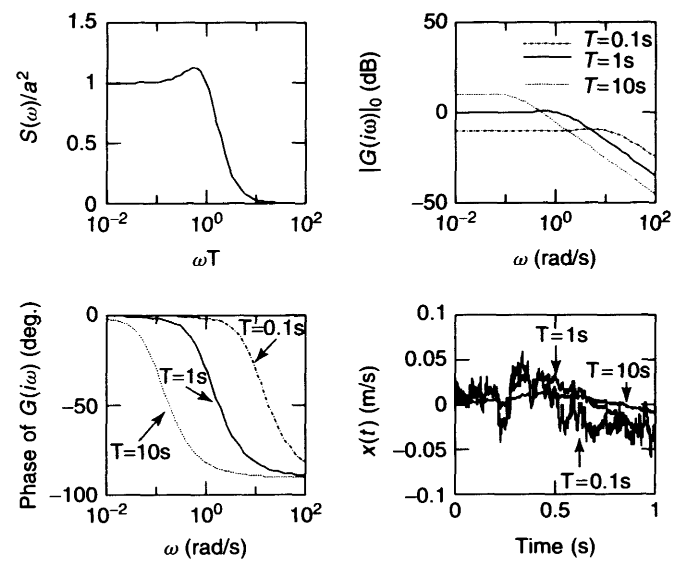

Figure 7.6 Power spectrum, white noise filter Bode plots, and simulated gust vertical velocity of the Dryden turbulence model