Finally, we calculate the estimated state-vector, $\mathbf{x_o}(t)$, by solving the Kalman filter state-equation (Eq. (7.44) with the known input, $u(t)$, and the measured output vector, $\mathbf{y}(t)$, using the SIMULINK block-diagram shown in Figure 7.8. The input, $u(t)$, is calculated in MATLAB work-space as follows, at the *same* time points previously used for generating the state-vector, $\mathbf{x}(t)$, with *ode45*:

```
>>u = 0.01*sin(5*t'); <enter>
```

The *actual* (i.e. generated by solving nonlinear equations through *ode45*) and estimated (SIMULINK) state variables are compared in Figure 7.8. Note that the state variables $\theta_1(t)$ and $\theta_1^{(1)}(t)$ are almost exactly estimated, as expected, because these state variables are directly measured. The estimation errors for $\theta_2(t)$ and $\theta_2^{(1)}(t)$ are appreciable, but reasonable, since we are trying to estimate a nonlinear plant by a linear Kalman filter.

7.5 Optimal (Linear, Quadratic, Gaussian) Compensators

In Chapter 5, we had used the *separation principle* to separately design a regulator and an observer using pole-placement, and put them together to form a *compensator* for the plant whose state-vector was unmeasurable. In Chapter 6, we presented *optimal control* techniques for designing linear regulators for multi-input plants that minimized a quadratic objective function, which included transient, terminal, and control penalties. In the present chapter, we have introduced the Kalman filter, which is an *optimal* observer for multi-output plants in the presence of process and measurement noise, modeled as white noises. Therefore, using a separation principle similar to that of Chapter 5, we can combine the optimal regulator of Chapter 6 with the optimal observer (the Kalman filter), and end up with an *optimal compensator* for multivariable plants. Since the optimal compensator is based upon a linear plant, a quadratic objective function, and an assumption of white noise that has a *normal*, or *Gaussian*, probability distribution, the optimal compensator is popularly called the *Linear, Quadratic, Gaussian* (or LQG) compensator. In short, the optimal compensator design process is the following:

(a) Design an *optimal* regulator for a linear plant assuming *full-state feedback* (i.e. assuming all the state variables are available for measurement) and a quadratic objective function (such as that given by Eq. (6.3)). The regulator is designed to generate a control input, $\mathbf{u}(t)$, based upon the measured state-vector, $\mathbf{x}(t)$.

(b) Design a Kalman filter for the plant assuming a known control input, $\mathbf{u}(t)$, a measured output, $\mathbf{y}(t)$, and white noises, $\mathbf{v}(t)$ and $\mathbf{z}(t)$, with known power spectral densities. The Kalman filter is designed to provide an *optimal estimate* of the state-vector, $\mathbf{x_o}(t)$.

(c) Combine the separately designed optimal regulator and Kalman filter into an optimal compensator, which generates the input vector, $\mathbf{u}(t)$, based upon the estimated state-vector, $\mathbf{x_o}(t)$, rather than the actual state-vector, $\mathbf{x}(t)$, and the measured output vector, $\mathbf{y}(t)$.

Since the optimal regulator and Kalman filter are designed separately, they can be selected to have desirable properties that are independent of one another. The closed-loop eigenvalues consist of the regulator eigenvalues and the Kalman filter eigenvalues, as seen in Chapter 5. The block diagram and state-equations for the closed-loop system with optimal compensator are the same as those for the pole-placement compensator designed in Chapter 5, except that now the plant contains process and measurement noise. The closed-loop system's performance can be obtained as desired by suitably selecting the optimal regulator's weighting matrices, $\mathbf{Q}$ and $\mathbf{R}$, and the Kalman filter's spectral noise densities, $\mathbf{V}$, $\mathbf{Z}$, and Ψ. Hence, the matrices $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{V}$, $\mathbf{Z}$, and Ψ are the *design parameters* for the closed-loop system with an optimal compensator.

A state-space realization of the optimal compensator for regulating a noisy plant with state-space representation of Eqs. (7.40) and (7.41) is given by the following state and output equations:

$$\mathbf{x}_o^{(1)}(t) = (\mathbf{A} - \mathbf{BK} - \mathbf{LC} + \mathbf{LDK})\mathbf{x}_o(t) + \mathbf{Ly}(t) \tag{7.74}$$

$$\mathbf{u}(t) = -\mathbf{Kx}_o(t) \tag{7.75}$$

where $\mathbf{K}$ and $\mathbf{L}$ are the optimal regulator and Kalman filter gain matrices, respectively. For a corresponding optimal tracking system, the state and output equations derived in Section 5.4.1 should be used with the understanding that $\mathbf{K}$ is the optimal feedback gain matrix and $\mathbf{L}$ is the optimal Kalman filter gain matrix.

Using MATLAB (CST), we can construct a state-space model of the regulating closed-loop system, *sysCL*, as follows:

```
>>sysp=ss(A,B,C,D); sysc=ss(A-B*K-L*C+L*D*K,L,K,zeros(size(D'))); <enter>

>>sysCL=feedback(sysp,sysc) <enter>
```

where *sysp* is the state-space model of the plant, and *sysc* is the state-space model of the LQG compensator. The resulting closed-loop system's state-vector is $\mathbf{x}_{\mathbf{CL}}(t) = [\mathbf{x}^T(t); \mathbf{x}_o{}^{\mathbf{T}}(t)]^T$. Alternatively, MATLAB (CST) provides a readymade command *reg* to construct a state-space model of the optimal compensator, given a state-space model of the plant, *sysp*, the optimal regulator feedback gain matrix, $\mathbf{K}$, and the Kalman filter gain matrix, $\mathbf{L}$. This command is used as follows:

```
>>sysc= reg(sysp,K,L) <enter>
```

where *sysc* is the state-space model of the compensator. To find the state-space representation of the closed-loop system, *sysCL*, the command *reg* should be followed by the command *feedback* as shown above.

Example 7.8

Let us design an optimal compensator for the flexible bomber aircraft (Examples 4.7, 6.1, 6.5, 6.7), with the process noise coefficient matrix, $\mathbf{F} = \mathbf{B}$. Recall that the sixth order, two input system is described by a linear, time-invariant, state-space representation given by Eq. (4.71). The inputs are the desired elevator deflection (rad.), $u_1(t)$, and the desired canard deflection (rad.), $u_2(t)$, while the outputs are the normal acceleration (m/s^2), $y_1(t)$, and the pitch-rate (rad./s), $y_2(t)$. In Example 6.1, we designed an optimal regulator for this plant to achieve a maximum overshoot of less than ± 2 m/s^2 in the normal-acceleration, and less than ± 0.03 rad/s in pitch-rate, and a settling time less than 5 s, while requiring elevator and canard deflections *not exceeding* ± 0.1 rad. (5.73°), if the initial condition is 0.1 rad/s perturbation in the pitch-rate, i.e. $\mathbf{x}(0) = [0; 0.1; 0; 0; 0; 0]^{\mathbf{T}}$. This was achieved with $\mathbf{Q} = 0.01\mathbf{I}$ and $\mathbf{R} = \mathbf{I}$, resulting in the following optimal feedback gain matrix:

$$\mathbf{K} = \begin{bmatrix} 1.0780 & -0.16677 & -0.046948 & -0.075618 & 0.59823 & 0.35302 \\ 1.3785 & 0.34502 & -0.013144 & -0.065260 & 0.47069 & 0.30941 \end{bmatrix}, \tag{7.76}$$

and the following eigenvalues of the regulator (i.e. eigenvalues of **A-BK**):

```
>>eig(A-B*K) <enter>
ans =
-7.8748e+001+ 5.0625e+001i
-7.8748e+001- 5.0625e+001i
-9.1803e+001
-1.1602e+000+ 1.7328e+000i
-1.1602e+000- 1.7328e+000i
-1.0560e+000
```

Note that the dominant regulator pole is at $s = -1.056$, which determines the *speed of response* of the full-state feedback control system. Since the closed-loop eigenvalues of the compensated system are the eigenvalues of the regulator and the eigenvalues of the Kalman filter, if we wish to achieve the same performance in the compensated system as the full-state feedback system, ideally we must select a Kalman filter such that the Kalman filter eigenvalues *do not dominate* the closed-loop system, i.e. they *should not* be closer to the imaginary axis than the regulator eigenvalues. As the Kalman filter does not require a control input, its eigenvalues can be pushed deeper into the left-half plane without causing concern of large required control effort (as in the case of the regulator). In other words, the Kalman filter can have *faster dynamics* than the regulator, which is achieved *free of cost.* However, as we will see in the present example, it is not always possible to push *all* the Kalman filter poles deeper into the left-half plane than the regulator poles by varying the noise spectral densities of the Kalman filter. In such cases, a judicious choice of Kalman filter spectral densities would yield the *best recovery of the full-state feedback dynamics.*

To design the Kalman filter to *recover* the full-state feedback performance of Example 6.1, the process noise spectral density matrix for the bomber is selected after some trial and error to be $\mathbf{V} = 0.0007\mathbf{B}^{\mathbf{T}}\mathbf{B}$, while the spectral density matrix of the measurement noise is $\mathbf{Z} = \mathbf{C}\mathbf{C}^{\mathbf{T}}$. We also assume that the process and measurement noises are uncorrelated, i.e. $\Psi = \mathbf{0}$. Then the Kalman filter gain is calculated using the MATLAB command *lqe* as follows:

```
>>[L,P,E] = lqe(A,B,C,0.0007*B'*B,C*C') <enter>
L =
-1.8370e-005  5.1824e-001
 9.8532e-004  7.0924e+000

-1.5055e-001 -2.6107e+001
-3.1291e-002 -6.7694e+001
-3.0779e-002 -1.0403e+001
-3.3715e-002  1.7954e-001

P =
 1.2307e+000 5.2093e-001 -3.9401e+001 5.5461e+001 -1.8815e-001 2.7655e-002
 5.2093e-001 6.9482e+000 -4.0617e+000 -6.3112e+001 -5.8958e+000 5.1163e+000
-3.9401e+001 -4.0617e+000 2.8063e+004 4.7549e+004 -2.7418e+002 -2.3069e+002
 5.5461e+001 -6.3112e+001 4.7549e+004 6.0281e+005 -1.1015e+003 -1.7972e+003
-1.8815e-001 -5.8958e+000 -2.7418e+002 -1.1015e+003 1.2093e+002 -2.4702e+001
 2.7655e-002 5.1163e+000 -2.3069e+002 -1.7972e+003 -2.4702e+001 3.2603e+002

E =
-1.4700e+002
-8.9112e+001
-7.9327e+000
-1.9548e+000+ 4.1212e+000i
-1.9548e+000- 4.1212e+000i
-5.6453e-001
```

Note that the Kalman filter's dominant pole is at $s = -0.56453$, which is closer to the imaginary axis than the dominant regulator pole. Thus, the two closed-loop

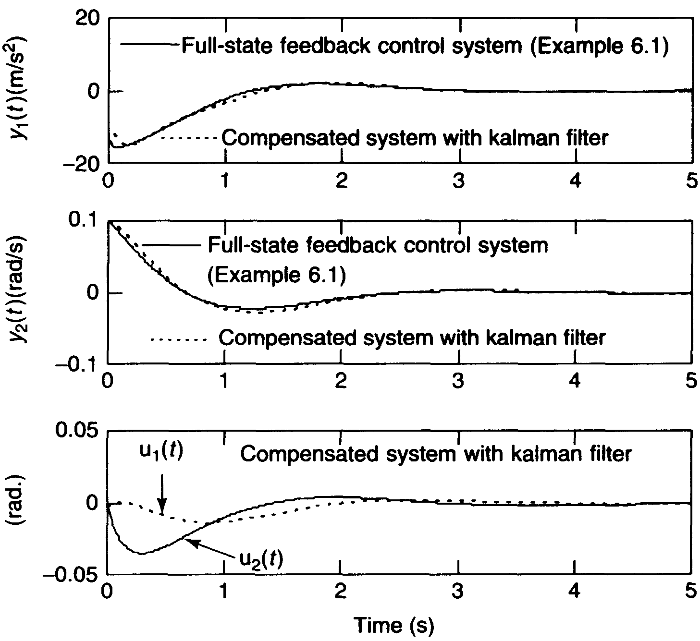

Figure 7.9 Closed-loop system's initial response and control inputs for the flexible bomber with optimal (LQG) compensator

poles *closest to the imaginary axis* are the dominant pole the Kalman filter and that of the optimal regulator, and the closed-loop response's speed and overshoots are largely determined by these two poles. We can construct a state-space model of the closed-loop system as follows:

```
>>sysp=ss(A,B,C,D); sysc=ss(A-B*K-L*C+L*D*K,L,K,zeros(size (D')));
  <enter>

>>sysCL=feedback(sysp,sysc) <enter>
```

You may confirm that the closed-loop eigenvalues are indeed the eigenvalues of the regulator and the Kalman filter using *damp(sysCL)*. The initial response of the closed-loop system to $\mathbf{x}(0) = [0; 0.1; 0; 0; 0; 0]^T$ and the required control input vector are calculated as follows:

```
>>[y,t,X]=initial(sysCL,[0 0.1 zeros(1,10)]'); u=-K* X(:,7:12)'; <enter>
```

The closed-loop responses, $y_1(t)$ and $y_2(t)$, are compared to their corresponding values for the full-state feedback control system (Example 6.1) in Figure 7.9. Figure 7.9 also contains the plots of the required control inputs, $u_1(t)$ and $u_2(t)$. Note that the speed of the response, indicated by the settling time of less than five seconds, maximum overshoots, and control input magnitudes are all quite similar to those of the full-state feedback system. This is the best recovery of the full-state feedback performance obtained by varying the Kalman filter's process noise spectral density scaling parameter, ρ, where $\mathbf{V} = \rho\mathbf{B}^{\mathbf{T}}\mathbf{B}$.

In Example 7.8, we saw how the full-state feedback control system's performance can be *recovered* by properly designing a Kalman filter to estimate the state-vector in the compensated system. In other words, the Kalman filter part of the optimal (LQG) compensator can be designed to yield approximately the same performance as that of the full-state feedback regulator. Now let us examine the robustness of the designed LQG compensated closed-loop system with respect to measurement noise in comparison with the full-state feedback system of Example 6.1. Such a comparison is valid, because both control systems use the same feedback gain matrix, $\mathbf{K}$. A SIMULINK block-diagram and the simulated initial response of the compensated closed-loop system are shown in Figure 7.10. The same white noise intensity (*power* parameter of 10^{-8}) in the pitch-rate channel is used as in Example 6.1. Comparing Figure 7.10 with Figure 6.3, we observe that the normal acceleration and pitch-rate fluctuations are about half the magnitudes and smaller in frequency than those seen in Figure 6.3. This indicates that the LQG compensated system is more robust with respect to measurement noise than the full-state feedback system of the same regulator gain matrix, $\mathbf{K}$.

We have thus far confined our attention to white noise model of disturbances. How robust would an LQG compensator be to *actual* parameter variations and noise which is not white? This is a crucial question, which is answered in the next section.