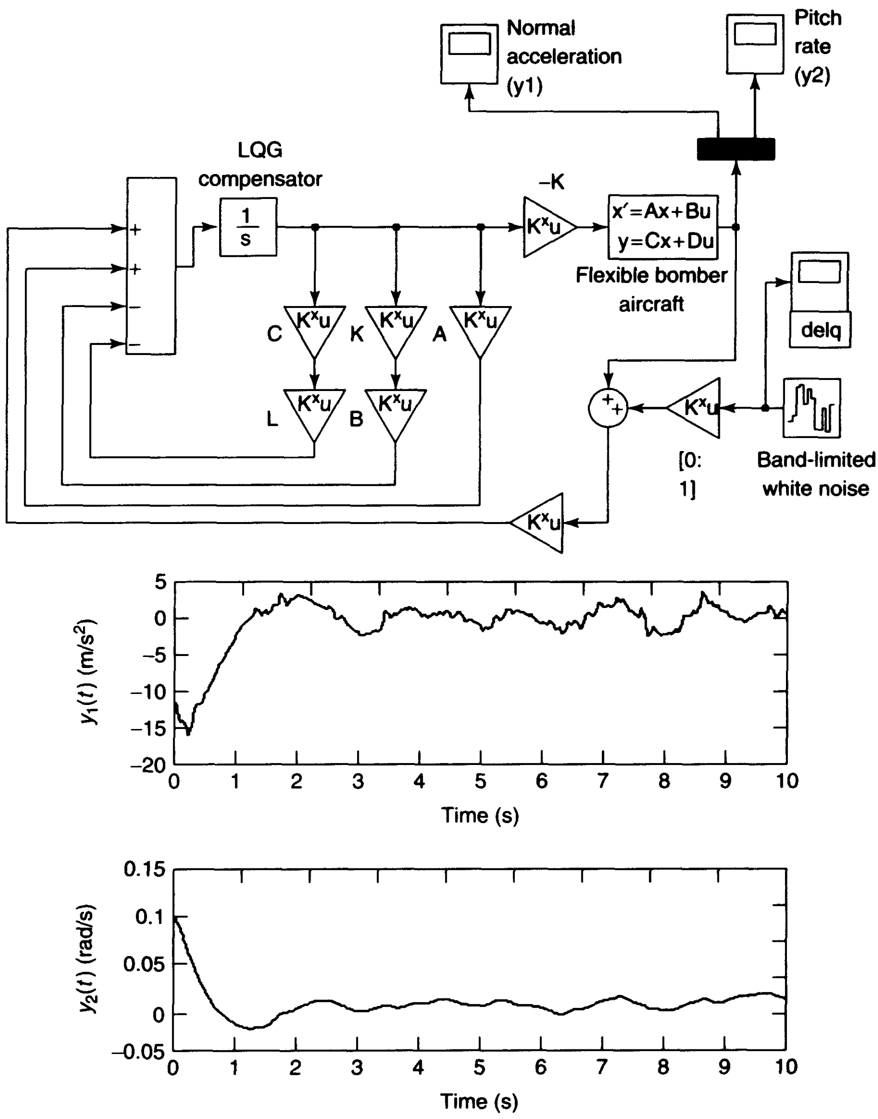

Figure 7.10 Simulated initial response of the flexible bomber with an optimal (LQG) compensator and measurement noise in the pitch-rate channel

7.6 Robust Multivariable LQG Control: Loop Transfer Recovery

In Chapter 2, we observed that in principle, the closed-loop systems are *less sensitive* (or *more robust*) to variations in the mathematical model of the plant (called *process noise and measurement noise* in Section 7.5), when compared to the corresponding open-loop systems. It was also observed that the *robustness* of a single-input, single-output feedback

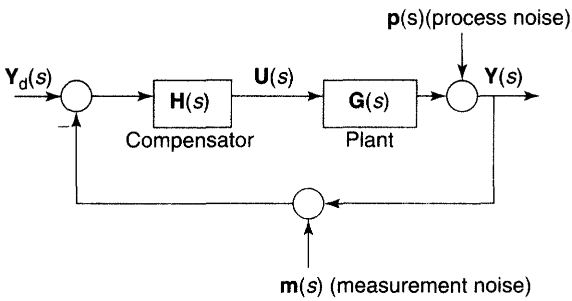

Figure 7.11 A multivariable feedback control system with compensator transfer matrix, $\mathbf{H}(s)$, plant transfer matrix, $\mathbf{G}(s)$, process noise, $\mathbf{p}(s)$, and measurement noise, $\mathbf{m}(s)$

control system is related to the *return difference*, $1 + G(s)H(s)$, where $G(s)$ and $H(s)$ are the transfer functions of the plant and controller, respectively. The larger the return difference of the feedback loop, the greater will be the robustness when compared to the corresponding open-loop system.

Consider a linear, time-invariant, multivariable feedback control system with the block-diagram shown in Figure 7.11. The control system consists of a feedback controller with transfer matrix, $\mathbf{H}(s)$, and a plant with transfer matrix, $\mathbf{G}(s)$, with desired output, $\mathbf{Y}_d(s)$. The *process noise*, $\mathbf{p}(s)$, and *measurement noise*, $\mathbf{m}(s)$, are present in the control system as shown. Using Figure 7.11, the output, $\mathbf{Y}(s)$, can be expressed as follows:

$$\begin{aligned}\mathbf{Y}(s) &= [\mathbf{I} + \mathbf{G}(s)\mathbf{H}(s)]^{-1}\mathbf{p}(s) - [\mathbf{I} - \{\mathbf{I} + \mathbf{G}(s)\mathbf{H}(s)\}^{-1}]\mathbf{m}(s)\\ &\quad + \{\mathbf{I} - [\mathbf{I} + \mathbf{G}(s)\mathbf{H}(s)]^{-1}\}\mathbf{Y}_d(s) \qquad (7.77)\end{aligned}$$

while the control input, $\mathbf{U}(s)$, can be expressed as

$$\begin{aligned}\mathbf{U}(s) &= [\mathbf{I} + \mathbf{H}(s)\mathbf{G}(s)]^{-1}\mathbf{H}(s)\mathbf{Y}_d(s) - [\mathbf{I} + \mathbf{H}(s)\mathbf{G}(s)]^{-1}\mathbf{H}(s)\mathbf{p}(s)\\ &\quad - [\mathbf{I} + \mathbf{H}(s)\mathbf{G}(s)]^{-1}\mathbf{H}(s)\mathbf{m}(s) \qquad (7.78)\end{aligned}$$

From Eqs. (7.77) and (7.78), it is clear that the *sensitivity* of the *output* with respect to process and measurement noise depends upon the matrix $[\mathbf{I} + \mathbf{G}(s)\mathbf{H}(s)]^{-1}$, while the sensitivity of the input to process and measurement noise depends upon the matrix $[\mathbf{I} + \mathbf{H}(s)\mathbf{G}(s)]^{-1}$. The larger the elements of these two matrices, the larger will be the sensitivity of the output and input to process and measurement noise. Since *robustness* is inversely proportional to *sensitivity*, we can extend the analogy to multivariable systems by saying that the *robustness of the output* is measured by the matrix $[\mathbf{I} + \mathbf{G}(s)\mathbf{H}(s)]$, called the *return-difference matrix at the output*, and the *robustness of the input* is measured by the matrix $[\mathbf{I} + \mathbf{H}(s)\mathbf{G}(s)]$, called the *return difference matrix at the plant input*. Therefore, for multivariable control-systems, there are *two* return difference matrices to be considered: the return difference at the output, $[\mathbf{I} + \mathbf{G}(s)\mathbf{H}(s)]$, and that at the plant's input, $[\mathbf{I} + \mathbf{H}(s)\mathbf{G}(s)]$. Alternatively, we can define the *return ratio matrices at the plant's output and input*, as $\mathbf{G}(s)\mathbf{H}(s)$ and $\mathbf{H}(s)\mathbf{G}(s)$, respectively, and measure robustness properties in terms of the *return ratios* rather than the *return differences*.

Continuing our analogy with single-input, single-output systems, we would like to assign a *scalar measure* to robustness, rather than deal with the two return-difference (or return ratio) matrices. We can define a *matrix norm* (introduced in Chapter 4) to assign a scalar measure to a matrix. For a complex matrix, **M**, with n rows and m columns, one such norm is the *spectral* (or *Hilbert*) norm, given by

$$\|\mathbf{M}\|_s = \sigma_{\max} \tag{7.79}$$

where $\sigma_{\max}$ is the *positive square-root of the maximum eigenvalue* of the matrix $\mathbf{M^H M}$ if $n \geq m$, or of the matrix $\mathbf{MM^H}$ if $n \leq m$. Here $\mathbf{M^H}$ denotes the *Hermitian of* **M** defined as the *transpose of the complex conjugate of* **M**. In MATLAB, the Hermitian of a *complex* matrix, **M**, is calculated by M', i.e. the same command as used for evaluating the *transpose* of *real* matrices. All positive square-roots of the eigenvalues of $\mathbf{M^H M}$ if $n \geq m$ (or $\mathbf{MM^H}$ if $n \leq m$) are called the *singular values of* **M**, and are denoted by σ_k, $k = 1, 2, .., n$, where n is the size of **M**. The largest among σ_k is denoted by $\sigma_{\max}$, and the smallest among σ_k is denoted by $\sigma_{\min}$. If **M** varies with frequency, then each singular value also varies with frequency.

A useful algorithm for calculating singular values of a complex matrix, **M**, of n rows and m columns is the *singular value decomposition*, which expresses **M** as follows:

$$\mathbf{M} = \mathbf{USV^H} \tag{7.80}$$

where **U** and **V** are complex matrices with the property $\mathbf{U^H U} = \mathbf{I}$ and $\mathbf{V^H V} = \mathbf{I}$, and **S** is a *real matrix* containing the singular values of **M** as the *diagonal elements of a square sub-matrix* of size ($n \times n$) or ($m \times m$), whichever is *smaller*. MATLAB also provides the function *svd* for computing the singular values by singular value decomposition, and is used as follows:

```
>>[U,S,V] = svd(M) <enter>
```

Example 7.9

Find the singular values of the following matrix:

$$\mathbf{M} = \begin{bmatrix} 1+i & 2+2i & 3+i \\ 1-i & 2-2i & 3-5i \\ 5 & 4+i & 1+2i \\ 7-i & 2 & 3+3i \end{bmatrix} \tag{7.81}$$

Using the MATLAB command *svd*, we get the matrices **U**, **S**, **V** of the singular value decomposition (Eq. (7.80)) as follows:

```
>>[U,S,V] = svd(M) <enter>

U =
 0.2945+0.0604i   0.4008-0.2157i    -0.0626-0.2319i   -0.5112-0.6192i
 0.0836-0.4294i   -0.0220-0.7052i   0.2569+0.1456i    0.4269-0.2036i
 0.5256+0.0431i   -0.3392-0.0455i   -0.4622-0.5346i   0.3230+0.0309i
 0.6620-0.0425i   -0.3236+0.2707i   0.5302+0.2728i    -0.1615-0.0154i
```

```
S =
 12.1277  0       0
     0    5.5662  0
     0    0       2.2218
     0    0       0

V =
0.6739            -0.6043            0.4251
0.4292-0.1318i    -0.0562-0.3573i    -0.7604-0.2990i
0.4793-0.3385i     0.6930-0.1540i     0.2254+0.3176i
```

Hence, the singular values of $\mathbf{M}$ are the diagonal elements of the (3×3) submatrix of $\mathbf{S}$, i.e. $\sigma_1(\mathbf{M}) = 2.2218$, $\sigma_2(\mathbf{M}) = 5.5662$, and $\sigma_3(\mathbf{M}) = 2.1277$, with the *largest* singular value, $\sigma_{\max}(\mathbf{M}) = 12.1277$ and the *smallest* singular value, $\sigma_{\min}(\mathbf{M}) = 2.2218$. Alternatively, we can directly use their definition to calculate the singular values as follows:

```
>>sigma= sqrt(eig(M'*M)) <enter>

sigma =
 12.1277+0.0000i
 5.5662+0.0000i
 2.2218+0.0000i
```

The singular values help us analyze the properties of a multivariable feedback (called a *multi-loop*) system in a manner quite similar to a single-input, single-output feedback (called a *single-loop*) system. For analyzing robustness, we can treat the *largest* and *smallest* singular values of a return difference (or return ratio) matrix as providing the *upper* and *lower* bounds on the *scalar* return difference (or return ratio) of an *equivalent* single-loop system. For example, to maximize robustness with respect to the process noise, it is clear from Eq. (7.77) that we should *minimize* the singular values of the *sensitivity matrix*, $[\mathbf{I} + \mathbf{G}(s)\mathbf{H}(s)]^{-1}$, which implies *minimizing* the *largest* singular value, $\sigma_{\max}[\{\mathbf{I} + \mathbf{G}(s)\mathbf{H}(s)\}^{-1}]$, or *maximizing* the singular values of the return difference matrix at the output, i.e. *maximizing* $\sigma_{\min}[\mathbf{I} + \mathbf{G}(s)\mathbf{H}(s)]$. The latter requirement is equivalent to *maximizing* the *smallest* singular value of the return ratio at the output, $\sigma_{\min}[\mathbf{G}(s)\mathbf{H}(s)]$. Similarly, *minimizing* the sensitivity to the measurement noise requires *minimizing* the *largest* singular value of the matrix $[\mathbf{I} - \{\mathbf{I} + \mathbf{G}(s)\mathbf{H}(s)\}^{-1}]$, which is equivalent to *minimizing* the *largest* singular value of the return ratio at the output, $\sigma_{\max}[\mathbf{G}(\mathrm{s})\mathbf{H}(\mathrm{s})]$. On the other hand, tracking a desired output requires from Eq. (7.77) that the *sensitivity* to $\mathbf{Y}_{\mathrm{d}}(s)$ be *maximized*, which requires *maximizing* $\sigma_{\min}[\mathbf{I} - \{\mathbf{I} + \mathbf{G}(s)\mathbf{H}(s)\}^{-1}]$, which is equivalent to maximizing $\sigma_{\min}[\mathbf{G}(s)\mathbf{H}(s)]$. Also, it is clear from Eq. (7.77) and the relationship $\mathbf{U}(s) = \mathbf{H}(s)\mathbf{Y}(s)$ that *optimal control* (i.e. minimization of control input magnitudes) requires *minimization* of $\sigma_{\max}[\mathbf{H}(s)\{\mathbf{I} + \mathbf{G}(s)\mathbf{H}(s)\}^{-1}]$, or alternatively, a minimization of $\sigma_{\max}[\mathbf{H}(s)]$. In summary, the following conditions on the singular values of the return ratio at output result from robustness, optimal control and tracking requirements:

(a) For *robustness* with respect to the *process noise*, $\sigma_{\min}[\mathbf{G}(s)\mathbf{H}(s)]$ should be *maximized.*

(b) For *robustness* with respect to the *measurement noise*, $\sigma_{\max}[\mathbf{G}(s)\mathbf{H}(s)]$ should be *minimized.*

(c) For *optimal control*, $\sigma_{\max}[\mathbf{H}(s)]$ should be *minimized.*

(d) For *tracking* a changing desired output, $\sigma_{\min}[\mathbf{G}(s)\mathbf{H}(s)]$ should be *maximized.*

Clearly, the second requirement conflicts with the first and the fourth. Also, since $\sigma_{\max}[\mathbf{G}(s)\mathbf{H}(s)] \le \sigma_{\max}[\mathbf{G}(s)]\sigma_{\max}[\mathbf{H}(s)]$ (a property of scalar norms), the third requirement is in conflict with the first and the fourth. However, since measurement noise usually has a predominantly high-frequency content (i.e. more peaks in the power spectrum at high frequencies), we achieve a compromise by minimizing $\sigma_{\max}[\mathbf{G}(s)\mathbf{H}(s)]$ (and $\sigma_{\max}[\mathbf{H}(s)]$) at *high frequencies*, and maximizing $\sigma_{\min}[\mathbf{G}(s)\mathbf{H}(s)]$ at *low frequencies*. In this manner, good robustness properties, optimal control, and tracking system performance can be obtained throughout a given frequency range.

The singular values of the return difference matrix at the output in the frequency domain, $\sigma[\mathbf{I} + \mathbf{G}(i\omega)\mathbf{H}(i\omega)]$, can be used to estimate the gain and phase margins (see Chapter 2) of a multivariable system. One can make singular value plots against frequency, ω, in a similar manner as the Bode gain plots. The way in which multivariable gain and phase margins are defined with respect to the singular values is as follows: take the *smallest* singular value, $\sigma_{\min}$, of all the singular values of the return difference matrix at the output, and find a real constant, a, such that $\sigma_{\min}[\mathbf{I} + \mathbf{G}(i\omega)] \ge a$ for all frequencies, ω, in the frequency range of interest. Then the gain and phase margins can be defined as follows:

$$\text{Gain margin} = 1/(1 \pm a) \tag{7.82}$$

$$\text{Phase margin} = \pm 2\sin^{-1}(a/2) \tag{7.83}$$

Example 7.10

Figure 7.12 shows the singular values of the rotating flexible spacecraft of Example 6.2. We observe that the smallest singular value reaches a minimum of -70 dB in the frequency range 0.01–10,000 rad/s at frequency 0.01 rad/s. Hence, $a = 10^{-(70/20)} = 3.16 \times 10^{-4}$. Therefore, the gain margins are $1/(1 + a) = 0.9997$ and $1/(1 - a) = 1.0003$, and phase margins are $\pm 0.0181°$ (which are quite small!). These margins are quite conservative, because they allow for *simultaneous* gain and phase variations of all the controller transfer functions. The present analysis indicates that the control system for the spacecraft cannot tolerate an appreciable variation in the phase of the return difference matrix before its eigenvalue cross into the right-half s-plane. However, the spacecraft is already *unstable* due to double eigenvalues at $s = 0$ (see Example 6.2), which the classical measures of gain and phase margins *do not* indicate (recall from Chapter 2 that gain and phase margins only indicate poles crossing over into right-half s-plane). Hence, gain and phase margins have limited utility for indicating closed-loop robustness.

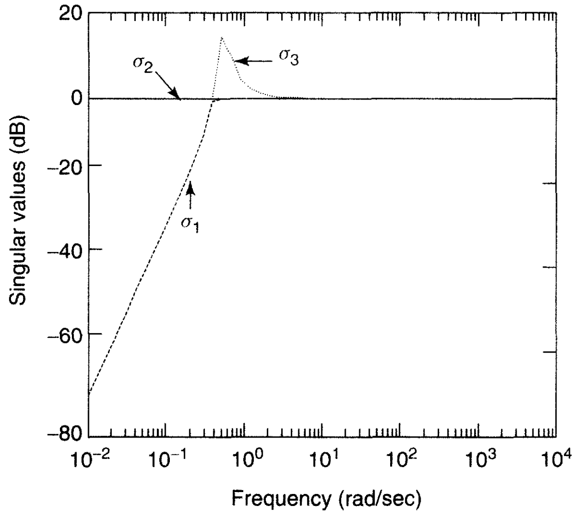

Figure 7.12 Singular value plot of the return difference matrix $[\mathbf{I} + \mathbf{G}(s)\mathbf{H}(s)]$ of the rotating spacecraft

For designing a control system in which the input vector is the *least sensitive* to process and measurement noise, we can derive appropriate conditions on the singular values of the return ratio at the plant's *input*, $\mathbf{H}(s)\mathbf{G}(s)$, by considering Eq. (7.78) as follows:

(a) For *robustness* with respect to the *process noise*, $\sigma_{\min}[\mathbf{H}(s)\mathbf{G}(s)]$ should be *maximized.*

(b) For *robustness* with respect to the *measurement noise*, $\sigma_{\min}[\mathbf{H}(s)\mathbf{G}(s)]$ should be *maximized.*

Thus, there is no conflict in achieving the robustness of the plant's input to process and measurement noise. However, Eq. (7.78) indicates that for tracking a changing desired output, $\sigma_{\max}[\mathbf{H}(s)\mathbf{G}(s)]$ should be *minimum*, which *conflicts with the robustness requirement.* This conflict can again be resolved by selecting different frequency ranges for maximizing and minimizing the singular values.

The adjustment of the singular values of return ratio matrices to achieve desired closed-loop robustness and performance is called *loop shaping.* This term is derived from single-loop systems where scalar return ratios of a loop are to be adjusted. For *compensated* systems based on an observer (i.e. the Kalman filter), generally there is a *loss* of robustness, when compared to *full-state feedback* control systems. To *recover* the robustness properties associated with full-state feedback, the Kalman filter must be designed such that the sensitivity of the *plant's input* to process and measurement noise is minimized. As seen above, this requires that the smallest singular value of the return ratio at plant's input, $\sigma_{\min}[\mathbf{H}(s)\mathbf{G}(s)]$, should be maximized. Theoretically, this maximum value of $\sigma_{\min}[\mathbf{H}(s)\mathbf{G}(s)]$ should be *equal* to that of the return ratio at the plant's input with full-state feedback. Such a process of designing a Kalman filter based compensator to recover the robustness of full-state feedback is called *loop transfer recovery*

(LTR). Optimal compensators designed with loop transfer recovery are called LQG/LTR compensators. The loop transfer recovery can either be conducted at the plant's the input, as described below, or at the plant's output. The design of optimal (LQG) compensators for loop transfer recovery at the plant's input can be stated as follows:

1. Design a full-state feedback optimal regulator by selecting $\mathbf{Q}$ and $\mathbf{R}$ matrices such that the desired performance objectives are met, and the singular values of the return ratio at the plant's input are maximized. With full-state feedback, the return ratio at the plant's input is $\mathbf{H}(s)\mathbf{G}(s) = -\mathbf{K}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$, where $\mathbf{A}$, $\mathbf{B}$ are the plant's state coefficient matrices and $\mathbf{K}$ is the full-state feedback regulator gain matrix.
2. Design a Kalman filter by selecting the noise coefficient matrix, $\mathbf{F}$, and the white noise spectral densities, $\mathbf{V}$, $\mathbf{Z}$, and Ψ, such that the singular values of the return ratio at the plant's input, $\mathbf{H}(s)\mathbf{G}(s)$, approach the corresponding singular values with full-state feedback. Hence, $\mathbf{F}$, $\mathbf{V}$, $\mathbf{Z}$, and Ψ are treated as *design parameters* of the Kalman filter to achieve full-state feedback return ratio at the plant's input, rather than *actual* parameters of process and measurement (white) noises.

The compensated system's return ratio matrix at plant's input can be obtained by taking the Laplace transform of Eqs. (7.74) and (7.75), and combining the results as follows:

$$\mathbf{U}(s) = -\mathbf{K}(s\mathbf{I} - \mathbf{A}_c)^{-1}\mathbf{L}\mathbf{Y}(s) \tag{7.84}$$

where $\mathbf{A}_c = (\mathbf{A} - \mathbf{BK} - \mathbf{LC} + \mathbf{LDK})$ is the compensator's state-dynamics matrix, $\mathbf{L}$ is the Kalman filter's gain matrix, and $\mathbf{Y}(s)$ is the plant's output, written as follows:

$$\mathbf{Y}(s) = [\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]\mathbf{U}(s) \tag{7.85}$$

Substituting Eq. (7.85) into Eq. (7.84), we get the following expression for $\mathbf{U}(s)$:

$$\mathbf{U}(s) = -\mathbf{K}(s\mathbf{I} - \mathbf{A}_c)^{-1}\mathbf{L}[\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]\mathbf{U}(s) \tag{7.86}$$

Note that the return ratio matrix at the plant's input is the matrix by which $\mathbf{U}(s)$ gets pre-multiplied in passing around the feedback loop and returning to itself, i.e. $\mathbf{U}(s) = \mathbf{H}(s)\mathbf{G}(s)$ in Figure 7.11 if all other *inputs* to the control system, $\mathbf{Y}_d(s)$, $\mathbf{p}(s)$, $\mathbf{m}(s)$, are zero. Hence, the return ratio at the compensated plant's input is

$$\mathbf{H}(s)\mathbf{G}(s) = -\mathbf{K}(s\mathbf{I} - \mathbf{A}_c)^{-1}\mathbf{L}[\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}] \tag{7.87}$$

There is no unique set of Kalman filter design parameters $\mathbf{F}$, $\mathbf{V}$, $\mathbf{Z}$, and Ψ to achieve loop transfer recovery. Specifically, if the plant is *square* (i.e. it has *equal* number of outputs and inputs) and *minimum-phase* (i.e. the plant's transfer matrix has *no zeros* in the right-half plane), then by selecting the noise coefficient matrix of the plant as $\mathbf{F} = \mathbf{B}$, the cross-spectral density as $\Psi = \mathbf{0}$, the measurement noise spectral density as $\mathbf{Z} = \mathbf{I}$, and the process noise spectral density as $\mathbf{V} = \mathbf{V}_o + \rho\mathbf{I}$, where ρ is a scaling parameter, it can be shown from the Kalman filter equations that in the limit $\rho \to \infty$, the compensated system's return ratio, given by Eq. (7.87), converges to $-\mathbf{K}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$, the return ratio of the full-state feedback system at the plant input. In most cases, *better* loop transfer recovery can be obtained by choosing $\mathbf{Z} = \mathbf{CC}^T$ and $\mathbf{V} = \rho\mathbf{B}^T\mathbf{B}$, and making ρ large. However, making ρ extremely large reduces the *roll-off* of the closed-loop transfer function at high frequencies, which is undesirable. Hence, instead of making ρ very large to achieve perfect loop transfer recovery at all frequencies, we should choose a value of ρ which is sufficiently large to *approximately* recover the return ratio over a given range of frequencies.

MATLAB's *Robust Control Toolbox* [3] provides the command *sigma* to calculate the singular values of a transfer matrix, $\mathbf{G}(s = i\omega)$, as a function of frequency, ω, as follows:

```
>>[sv,w] = sigma(sys) <enter>
```

where *sys* is an LTI object of $\mathbf{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$, and **sv** and **w** contain the returned singular values and frequency points, respectively. The user can specify the set of frequencies at which the singular values of $\mathbf{G}(s)$ are to be computed by including the frequency vector, **w**, as an additional input argument of the command as follows:

```
>>[sv,w] = sigma(sys,w) <enter>
```

Also, *sigma* can calculate the singular values of some commonly encountered functions of $\mathbf{G}(i\omega)$ by using the command as follows:

```
>>[sv,w] = sigma(sys,w,type) <enter>
```

where *type* = 1, 2, or 3 specify that singular values of $\mathbf{G}^{-1}(i\omega)$, $\mathbf{I} + \mathbf{G}(i\omega)$, or $\mathbf{I} + \mathbf{G}^{-1}(i\omega)$, respectively, are to be calculated. (Of course, *sigma* requires that $\mathbf{G}(i\omega)$ should be a square matrix.) Hence, *sigma* is a versatile command, and can be easily used to compute singular values of return ratios, or return difference matrices. If you do not have the *Robust Control Toolbox*, you can write your own M-file for calculating the singular value spectrum using the MATLAB functions *svd* or *eig* as discussed above.

Example 7.11

Re-consider the flexible bomber airplane of Example 7.8, where we designed an optimal compensator using an optimal regulator with $\mathbf{Q} = 0.01\mathbf{I}$, $\mathbf{R} = \mathbf{I}$, and a Kalman filter with $\mathbf{F} = \mathbf{B}$, $\mathbf{V} = 0.0007\mathbf{B}^T\mathbf{B}$, $\mathbf{Z} = \mathbf{CC}^T$, and $\Psi = \mathbf{0}$, to *recover* the *performance* of the full-state feedback regulator. Let us now see how *robust* such a compensator is by studying the return ratio at the plant's input. Recall that the return ratio at the plant's input is $\mathbf{H}(s)\mathbf{G}(s)$, which is the transfer matrix of a hypothetical system formed by placing the plant, $\mathbf{G}(s)$, in series with the compensator, $\mathbf{H}(s)$ (the plant *is followed* by the compensator). Hence, we can find a state-space representation of $\mathbf{H}(s)\mathbf{G}(s)$ in terms of the state-space model, *sysHG*, with plant model, *sysp*, and compensator model, *sysc*, as follows:

```
>>sysp=ss(A,B,C,D);sysc=ss(A-B*K-L*C+L*D*K,L,-K,zeros(size(K,1)));
  <enter>

>> sysHG = series(sysp,sysc); <enter>
```

The singular value spectrum of the return ratio at the plant's input for frequency range $10^{-2}-10^{4}$ rad/s is calculated as follows:

```
>>w = logspace(-2,4); [sv,w] = sigma(sysHG,w); <enter>
```

while the singular value spectrum of the return ratio at plant's input of the full-state feedback system is calculated in the same frequency range by

```
>>sysfs=ss(A,B,-K,zeros(size(K,1))); [sv1,w1] = sigma(sysfs,w);
  <enter>
```

The two sets of singular values, **sv** and **sv1**, are compared in Figure 7.13, which is plotted using the following command:

```
>>semilogx(w,20*log10(sv),':',w1,20*log10(sv1)) <enter>
```

Note that there is a large difference between the smallest singular values of the compensated and full-state feedback systems, indicating that the compensated system is *much less* robust than the full-state feedback system. For recovering the full-state feedback robustness at the plant's input, we re-design the Kalman filter using $\mathbf{V} = \rho\mathbf{B}^{\mathbf{T}}\mathbf{B}$, $\mathbf{Z} = \mathbf{C}\mathbf{C}^{\mathbf{T}}$, and $\Psi = \mathbf{0}$, where ρ is a scaling parameter for the process noise spectral density. As ρ is increased, say, from 10 to 10^{8}, the return ratio of the compensated plant approaches that of the full-state feedback system *over a larger range of frequencies*, as seen in the singular value plots of Figure 7.14. For $\rho = 10$, the smallest singular value of the compensated system's return ratio becomes equal to that of the full-state feedback system in the frequency range 1–100 rad/s, while for $\rho = 10^{8}$ the range of frequencies (or bandwidth) over which

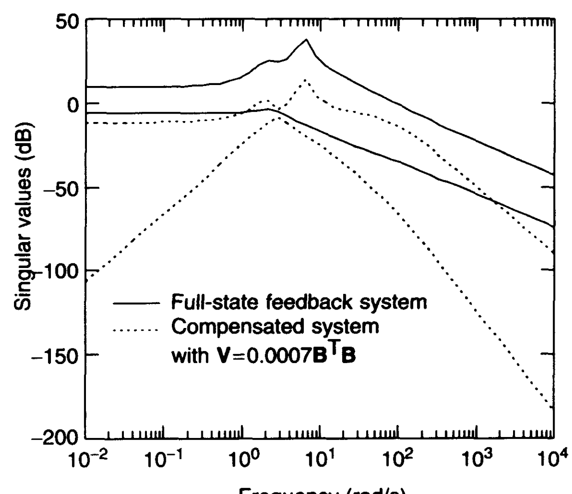

Figure 7.13 Singular values of the return ratio matrix, **H**(*s*)**G**(*s*), at the plant's input of compensated system and full-state feedback system for the flexible bomber

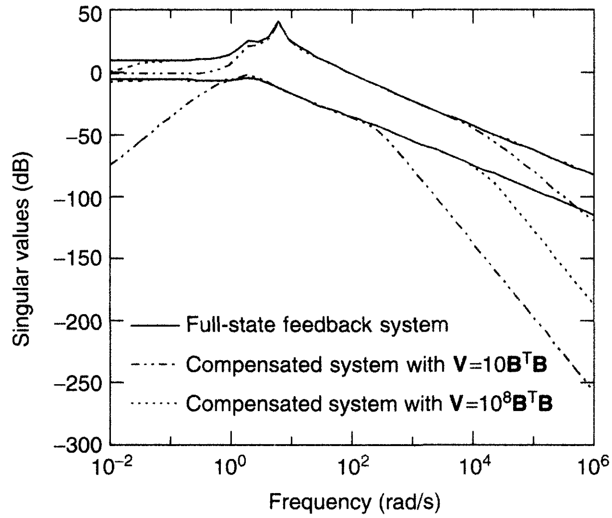

Figure 7.14 Comparison of singular values of return ratio matrix at the plant's input of full-state feedback system and compensated system for loop transfer recovery with process noise spectral density, $\mathbf{V} = 10\mathbf{B}^T\mathbf{B}$ and $\mathbf{V} = 10^8\mathbf{B}^T\mathbf{B}$ (flexible bomber)

the loop transfer recovery occurs increases to 10^{-2}–10^4 rad/s. At frequencies higher than the loop transfer recovery bandwidth the return ratio is seen to roll-off at more than 50dB/decade. Such a roll-off would also be present in the singular values of the closed-loop transfer matrix, which is good for rejection of noise at high frequencies. However, within the loop transfer recovery bandwidth, the roll-off is only about 20dB/decade. Therefore, the larger the bandwidth for loop transfer recovery, the smaller would be the range of frequencies over which high noise attenuation is provided by the compensator. Hence, the loop transfer recovery bandwidth must not be chosen to be too large; otherwise high frequency noise (usually the measurement noise) would get unnecessarily amplified by smaller roll-off provided within the LTR bandwidth.

Note that the Kalman filter designed in Example 7.8 with $\mathbf{V} = 0.0007\mathbf{B}^T\mathbf{B}$ recovers the *performance* of the full-state feedback system (with a loss of robustness), whereas the re-designed Kalman filter with $\mathbf{V} = 10^8\mathbf{B}^T\mathbf{B}$ recovers the *robustness* of the full-state feedback system over a bandwidth of 10^6 rad/s (with an expected loss of performance). By choosing a large value of the process noise spectral density for loop transfer recovery, a pair of Kalman filter poles comes very close to the imaginary axis and becomes the dominant pole configuration, thereby playing havoc with the performance. Hence, there is a *contradiction* in recovering *both* performance and robustness with the same Kalman filter, and a compromise must be made between the two.

It is interesting to note that Example 7.11 has a *non-minimum phase* plant, with a plant zero at $s = 2.5034 \times 10^{-7}$. The loop transfer recovery is *not guaranteed* for non-minimum phase plants, because it is pointed out above that $\mathbf{H}(s)\mathbf{G}(s)$ converges to $-\mathbf{K}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$ in the limit of *infinite* process noise spectral density, provided that the plant is *square* and *minimum phase*. The reason why LQG/LTR compensators generally cannot be designed for non-minimum phase plants is that perfect loop transfer recovery requires placing *some poles* of the Kalman filter at the *zeros* of the plant [4]. If the plant is *non-minimum phase*, it implies that the Kalman filter for perfect loop transfer recovery should be *unstable*. However, if the right-half plane zeros of a non-minimum phase plant are very close to the imaginary axis (as in Example 7.11), the frequency associated with it lies *outside* the selected bandwidth for loop transfer recovery, and hence loop transfer recovery in a given bandwidth is still possible, as seen above.

Example 7.12

Let us design an optimal LQG/LTR compensator for the flexible, rotating spacecraft of Example 6.2. The spacecraft consists of a rigid hub and four flexible appendages, each having a tip mass, with three torque inputs in N-m, $u_1(t)$, $u_2(t)$, $u_3(t)$, and three angular rotation outputs in rad., $y_1(t)$, $y_2(t)$, $y_3(t)$. A linear, time-invariant state-space representation of the 26th order spacecraft was given in Example 6.2, where it was observed that the spacecraft is *unstable* and *uncontrollable*. The natural frequencies of the spacecraft, including structural vibration frequencies, range from 0–47 588 rad/s. The *uncontrollable* modes are the structural vibration modes, while the *unstable* mode is the rigid-body rotation with zero natural frequency. Hence, the spacecraft is *stabilizable* and an optimal regulator with $\mathbf{Q} = 200\mathbf{I}$, and $\mathbf{R} = \mathbf{I}$ was designed in Example 6.2 to stabilize the spacecraft, with a settling time of 5 s, with zero maximum overshoots, while requiring input torques not exceeding 0.1 N-m, when the spacecraft is initially perturbed by a hub rotation of 0.01 rad. (i.e. the initial condition is $\mathbf{x}(0) = [0.01;\ \text{zeros}(1, 25)]^T$).

We would like to combine the optimal regulator already designed in Example 6.2 with a Kalman filter that recovers the return ratio at the plant's input in the frequency range 0–50 000 rad/s (approximately the bandwidth of the plant). To do so, we select $\mathbf{F} = \mathbf{B}$, $\mathbf{Z} = \mathbf{C}\mathbf{C}^{\mathbf{T}}$, $\Psi = \mathbf{0}$, and $\mathbf{V} = \rho\mathbf{B}^{\mathbf{T}}\mathbf{B}$, where ρ is a scaling parameter. By comparing the singular values of $\mathbf{H}(s)\mathbf{G}(s)$ with those of $-\mathbf{K}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$, for various values of ρ, we find that loop transfer recovery occurs in the desired bandwidth for $\rho = 10^{22}$, for which the Kalman filter gain, covariance, and eigenvalues are obtained as follows (only eigenvalues are shown below):

```
>>[L,P,E] = lqe(A,B,C,1e12*B'*B,C*C') <enter>

E =
-1.0977e+008+ 1.0977e+008i
-1.0977e+008- 1.0977e+008i
-1.0977e+008+ 1.0977e+008i
-1.0977e+008- 1.0977e+008i
-8.6042e+004+ 8.6074e+004i
-8.6042e+004- 8.6074e+004i
-4.6032e+000+ 4.5876e+004i
-4.6032e+000- 4.5876e+004i
```

```
-8.7040e-001+ 2.3193e+004i
-8.7040e-001- 2.3193e+004i
-3.4073e-001+ 1.0077e+004i
-3.4073e-001- 1.0077e+004i
-3.5758e-002+ 3.7121e+003i
-3.5758e-002- 3.7121e+003i
-1.0975e-002+ 4.0386e+002i
-1.0975e-002- 4.0386e+002i
-1.2767e-004+ 4.5875e+004i
-1.2767e-004- 4.5875e+004i
-9.3867e-005+ 3.7121e+003i
-9.3867e-005- 3.7121e+003i
-7.4172e-005+ 1.0077e+004i
-7.4172e-005- 1.0077e+004i
-5.8966e-005+ 2.3193e+004i
-5.8966e-005- 2.3193e+004i
-3.8564e-005+ 4.0385e+002i
-3.8564e-005- 4.0385e+002i
```

Note that the Kalman filter is stable, but has some eigenvalues very close to the imaginary axis, which is likely to degrade the performance of the closed-loop system. The singular values of the return ratio at the plant's input, for the compensated system are calculated as follows, with **K** obtained in Example 6.2:

```
>>sysp=ss(A,B,C,D);sysc=ss(A-B*K-L*C+L*D*K,L,-K,zeros(size(K,1)));
  <enter>

>> sysHG = series(sysp,sysc); <enter>

>>w = logspace(-2,6); [sv,w] = sigma(sysHG,w); <enter>
```

The singular values of the full-state feedback return ratio at the plant's input are obtained as follows:

```
>>sysfs=ss(A,B,-K,zeros(size(K,1))); [sv1,w1] = sigma(sysfs,w); <enter>
```

The two sets of singular values (full-state feedback and compensated system) are compared in Figure 7.15. Note that the smallest singular value of the return ratio is recovered in the range 0–50 000 rad/s, as desired, while the other two singular values are recovered in a larger frequency range. The plant has some zeros very close to the imaginary axis (largest real part is 3.3×10^{-8}) in the *right-half plane*, hence, the plant is *non-minimum phase* and perfect loop transfer recovery is not guaranteed. Also, due to this nature of the plant, the function *lqe* may yield inaccurate results.

Owing to the presence of some Kalman filter poles very close to the imaginary axis, the closed-loop system would have unacceptable performance. Hence, for improving the closed-loop performance, the loop transfer recovery bandwidth must be reduced. A good compromise between performance and robustness is achieved by choosing $\rho = 10^6$. Figure 7.16 shows the spectrum of the smallest singular value of

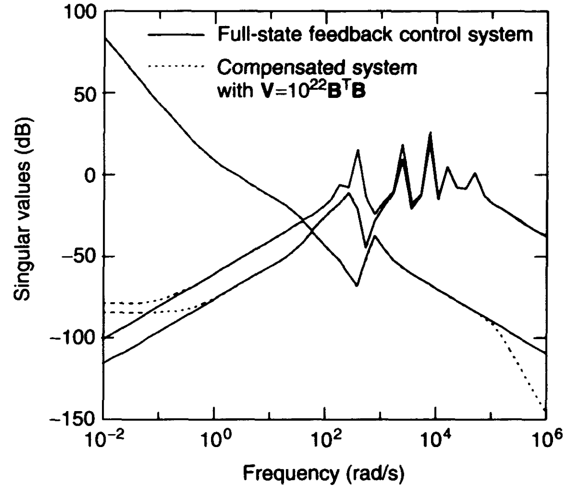

Figure 7.15 Singular values of the return ratio matrix at the plant's input of the full-state feedback system compared with those of the compensated plant with $\mathbf{V} = 10^{22}\mathbf{B}^T\mathbf{B}$ for the flexible, rotating spacecraft

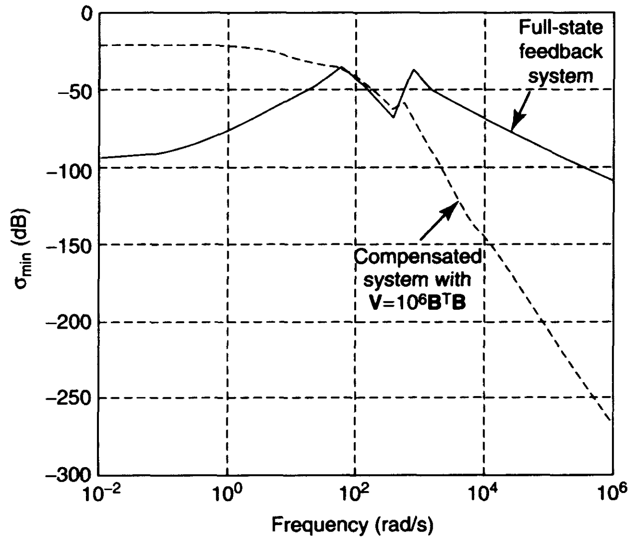

Figure 7.16 Smallest singular value of $\mathbf{H}(s)\mathbf{G}(s)$ of the optimal (LQG) compensated system with $\mathbf{V} = 10^6\mathbf{B}^T\mathbf{B}$ for the flexible rotating spacecraft, compared with that of the full-state feedback system showing loop-transfer recovery in the frequency range 5–500 rad/s

$\mathbf{H}(s)\mathbf{G}(s)$ which indicates a loop transfer recovery bandwidth of 50–500 rad/s. The performance of the closed-loop system is determined using a SIMULINK simulation of the initial response with $\mathbf{x}(0) = [0.01;\ zeros(1, 25)]^T$) and measurement noise in the hub-rotation angle, $y_1(t)$, as shown in Figure 7.17. A closed-loop settling time

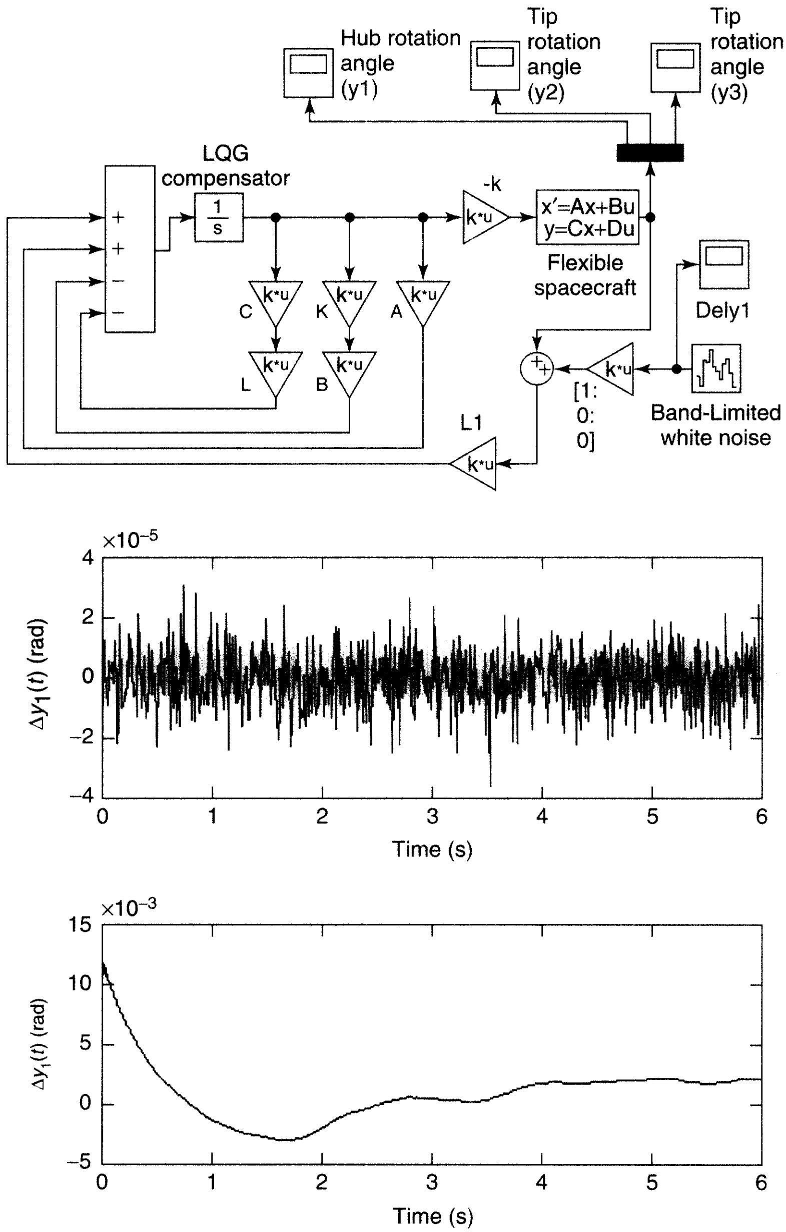

Figure 7.17 Simulated initial response of the LQG compensated flexible rotating spacecraft's hub-rotation angle, $y_1(t)$, with measurement noise, $\Delta y_1(t)$, in the hub-rotation channel

of five seconds, with maximum overshoots less than 1% and a steady-state error of 0.002 rad, is observed when the measurement error in $y_1(t)$ is $\pm 3.5 \times 10^{-5}$ rad. Comparing with the full-state feedback response with zero noise seen in Figure 6.5, the performance is fairly robust with respect to measurement noise.