some additional known, non-stochastic inputs of the plant (such as desired outputs) can be specified as an additional argument, *KNOWN*, while control inputs (i.e. inputs applied by the compensator to the plant) are specified by an additional argument, *CONTROLS*. The resulting compensator has control feedback commands as outputs and the known inputs and sensors as inputs. The command *dreg* is thus used in its most general form as follows:

```
>>[ac,bc,cc,dc] = dreg(Ad,Bd,Cd,Dd,K,Lstar,SENSORS,KNOWN,CONTROLS); <enter>
```

The greatest utility of *dreg* lies in the digital equivalent of LQG compensators. We will discuss digital optimal control in the next section.

8.11 Linear Optimal Control of Digital Systems

In a manner similar to the linear optimal control of analog systems presented in Chapter 6, we can devise equivalent techniques for linear optimal digital control. The digital optimal control expressions are in terms of *difference equations* rather than *differential equations* implementation presented in Chapter 6. We saw in the previous section how the implementation of a difference equation for a current digital observer resulted in a state-space model (Eq. (8.86)) that was quite different from that of the analog observer of Eq. (5.97). A similar modification is expected in the solution to the digital optimal control problem. Let us begin with the full-state feedback regulator design. An objective function to be minimized for the control of a linear, *time-varying* digital system can be expressed as the digital equivalent of Eq. (6.31) as follows:

$$J(0, N) = \sum_{k=0}^{N} \mathbf{x}^{\mathrm{T}}(k)\mathbf{Q}(k)\mathbf{x}(k) + \mathbf{u}^{\mathrm{T}}(k)\mathbf{R}(k)\mathbf{u}(k) \tag{8.94}$$

The *quadratic form* of the objective function, $J(0, N)$, given by Eq. (8.94) suffers no loss of generality if we assume both $\mathbf{Q}(k)$ and $\mathbf{R}(k)$ to be *symmetric* matrices. The *optimal* control gain matrix, $\mathbf{K}(k)$, of the control law, $\mathbf{u}(k) = -\mathbf{K}(k)\mathbf{x}(k)$, is to be chosen such that $J(0, N)$ is minimized *with respect to* the control input, $\mathbf{u}(k)$, *subject to* the constraint that the state-vector, $\mathbf{x}(k)$, is the solution of the following time-varying state-equation:

$$\mathbf{x}(k+1) = \mathbf{A}(k)\mathbf{x}(k) + \mathbf{B}(k)\mathbf{u}(k) \tag{8.95}$$

Note that we have dropped the subscript d from the coefficient matrices, $\mathbf{A}(k)$ and $\mathbf{B}(k)$, in Eq. (8.95) – indicating a digital system – for simplicity of notation. To derive the optimal regulator gain matrix which minimizes $J(0, N)$, it is more expedient to work with $J(m, N)$ which can be expressed as follows:

$$J(m, N) = J(0, N) - J(0, m-1) = F(m) + F(m+1) + \cdots + F(N-1) + F(N) \tag{8.96}$$

where $F(k) = \mathbf{x}^{\mathrm{T}}(k)\mathbf{Q}(k)\mathbf{x}(k) + \mathbf{u}^{\mathrm{T}}(k)\mathbf{R}(k)\mathbf{u}(k)$. To Eq. (8.96), we can apply the *principle of optimality* [2], which states that if the regulator, $\mathbf{u}(k) = -\mathbf{K}(k)\mathbf{x}(k)$, is optimal for the interval $0 \le k \le N$, then it is *also* optimal over *any* sub-interval, $m \le k \le N$, where $0 \le m \le N$. The principle of optimality can be applied by first minimizing $J(N, N) = F(N)$, and then finding $F(N-1)$ to minimize $J(N-1, N) = F(N-1) + F(N) = F(N-1) + J_o(N, N)$, where $J_o(N, N)$ refers to the minimum value of $J(N, N)$, and continuing this process until $J(0, N)$ is minimized. Such a sequential minimization is called *dynamic programming* in optimization parlance. Note that as in the case of analog optimal control (Chapter 6), the digital optimal control solution is obtained by marching *backwards* in time, beginning with the minimization of $J(N, N)$.

The minimum value of $J(N, N) = F(N) = \mathbf{x}^{\mathbf{T}}(N)\mathbf{Q}(N)\mathbf{x}(N) + \mathbf{u}^{\mathbf{T}}(N)\mathbf{R}(N)\mathbf{u}(N)$ with respect to the input, $\mathbf{u}(N)$, is easily obtained to be

$$J_o(N, N) = \mathbf{x}^{\mathbf{T}}(N)\mathbf{Q}(N)\mathbf{x}(N) \tag{8.97}$$

because the state at the final sampling instant, $\mathbf{x}(N)$, is *independent* of the final control input, $\mathbf{u}(N)$, and the minimum value of $\mathbf{u}^{\mathbf{T}}(N)\mathbf{R}(N)\mathbf{u}(N) = 0$, which occurs for the *optimal control input*, $\mathbf{u_o}(N) = \mathbf{0}$. Substituting Eq. (8.95) into Eq. (8.97) for $k = N - 1$, we get

$$\begin{aligned} J_o(N, N) = {} & [\mathbf{A}(N-1)\mathbf{x}(N-1) + \mathbf{B}(N-1)\mathbf{u}(N-1)]^T\mathbf{Q}(N-1)[\mathbf{A}(N-1)\mathbf{x}(N-1) \\ & + \mathbf{B}(N-1)\mathbf{u}(N-1)]_{\mathbf{u_o}(N-1)} \end{aligned} \tag{8.98}$$

where the subscript $\mathbf{u_o}(N-1)$ indicates that the expression on the right-hand side of Eq. (8.98) is evaluated for the optimal control input at the $(N-1)$th sampling instant. Then $J(N-1, N)$ is determined by substituting Eq. (8.98) into $J(N-1, N) = F(N-1) + J_o(N, N)$ to yield

$$\begin{aligned} J(N-1, N) = {} & \mathbf{x}^{\mathbf{T}}(N-1)\mathbf{Q}(N-1)\mathbf{x}(N-1) + \mathbf{u}^{\mathbf{T}}(N-1)\mathbf{R}(N-1)\mathbf{u}(N-1) \\ & + [\mathbf{A}(N-1)\mathbf{x}(N-1) + \mathbf{B}(N-1)\mathbf{u}(N-1)]^T\mathbf{Q}(N-1) \\ & \times [\mathbf{A}(N-1)\mathbf{x}(N-1) + \mathbf{B}(N-1)\mathbf{u}(N-1)]|_{\mathbf{u_o}(N-1)} \end{aligned} \tag{8.99}$$

The optimal control input at the $(N-1)$th sampling instant, $\mathbf{u_o}(N-1)$, can be evaluated by solving

$$\partial J(N-1, N)/\partial \mathbf{u}(N-1) = \mathbf{0} \tag{8.100}$$

Since $J(N-1, N)$ consists of quadratic terms, such as $\mathbf{u}^{\mathbf{T}}\mathbf{R}\mathbf{u}$, and *bilinear* terms, such as $\mathbf{u}^{\mathbf{T}}\mathbf{Q}\mathbf{x}$ and $\mathbf{x}^{\mathbf{T}}\mathbf{Q}\mathbf{u}$, to evaluate the derivative in Eq. (8.100), we must know how to differentiate such *scalar* terms with respect to the *vector*, $\mathbf{u}(N-1)$. From Appendix B, we can write $\partial(\mathbf{u}^{\mathbf{T}}\mathbf{R}\mathbf{u})/\partial\mathbf{u} = 2\mathbf{R}\mathbf{u}$, $\partial(\mathbf{u}^{\mathbf{T}}\mathbf{Q}\mathbf{x})/\partial\mathbf{u} = \mathbf{Q}\mathbf{x}$, and $\partial(\mathbf{x}^{\mathbf{T}}\mathbf{Q}\mathbf{u})/\partial\mathbf{u} = \mathbf{Q}^{\mathbf{T}}\mathbf{x}$. After carrying out the differentiation in Eq. (8.100), we get

$$\begin{aligned} & 2\mathbf{R}(N-1)\mathbf{u_o}(N-1) + 2\mathbf{B}^{\mathbf{T}}(N-1)\mathbf{Q}(N-1)\mathbf{B}(N-1)\mathbf{u_o}(N-1) \\ & + [\mathbf{A}^{\mathbf{T}}(N-1)\mathbf{Q}(N-1)\mathbf{B}(N-1)]^T\mathbf{x}(N-1) \\ & + \mathbf{B}^{\mathbf{T}}(N-1)\mathbf{Q}(N-1)\mathbf{A}(N-1)\mathbf{x}(N-1) = \mathbf{0} \end{aligned} \tag{8.101}$$

or

$$2\mathbf{R}(N-1)\mathbf{u_o}(N-1) + 2\mathbf{B}^{\mathbf{T}}(N-1)\mathbf{Q}(N-1)\mathbf{B}(N-1)\mathbf{u_o}(N-1) \\ + 2\mathbf{B}^{\mathbf{T}}(N-1)\mathbf{Q}(N-1)\mathbf{A}(N-1)\mathbf{x}(N-1) = \mathbf{0} \tag{8.102}$$

which yields

$$\mathbf{u_o}(N-1) = -[\mathbf{R}(N-1) + \mathbf{B}^{\mathbf{T}}(N-1)\mathbf{Q}(N-1)\mathbf{B}(N-1)]^{-1} \\ + \mathbf{B}^{\mathbf{T}}(N-1)\mathbf{Q}(N-1)\mathbf{A}(N-1)\mathbf{x}(N-1) \tag{8.103}$$

Comparing Eq. (8.103) with the control law $\mathbf{u}(k) = -\mathbf{K}(k)\mathbf{x}(k)$, the *optimal regulator gain matrix* at the $(N-1)$th sampling instant, $\mathbf{K_o}(N-1)$, is obtained to be

$$\mathbf{K_o}(N-1) = [\mathbf{R}(N-1) + \mathbf{B}^{\mathbf{T}}(N-1)\mathbf{Q}(N-1)\mathbf{B}(N-1)]^{-1} \\ \times \mathbf{B}^{\mathbf{T}}(N-1)\mathbf{Q}(N-1)\mathbf{A}(N-1) \tag{8.104}$$

Substituting Eq. (8.103) into Eq. (8.99), we get the following quadratic form for $J_o(N-1, N)$:

$$J_o(N-1, N) = \mathbf{x}^{\mathbf{T}}(N-1)\mathbf{M}(N-1)\mathbf{x}(N-1) \tag{8.105}$$

where $\mathbf{M}(N-1)$ is a symmetric matrix. Continuing in this manner, we can write

$$J_o(N-2, N) = \mathbf{x}^{\mathbf{T}}(N-2)\mathbf{M}(N-2)\mathbf{x}(N-2) \tag{8.106}$$

$$J_o(m-1, N) = \mathbf{x}^{\mathbf{T}}(m-1)\mathbf{M}(m-1)\mathbf{x}(m-1) \tag{8.107}$$

and

$$J_o(m, N) = \mathbf{x}^{\mathbf{T}}(m)\mathbf{M}(m)\mathbf{x}(m) \tag{8.108}$$

Substituting Eq. (8.95) into Eq. (8.108) for $k = m-1$, we can write

$$J_o(m, N) = [\mathbf{A}(m-1)\mathbf{x}(m-1) + \mathbf{B}(m-1)\mathbf{u}(m-1)]^T\mathbf{M}(m) \\ \times [\mathbf{A}(m-1)\mathbf{x}(m-1) + \mathbf{B}(m-1)\mathbf{u}(m-1)] \tag{8.109}$$

To obtain the optimal control input at the $(m-1)$th sampling instant, $\mathbf{u_o}(m-1)$, we must minimize $J(m-1, N)$, which is written as

$$\begin{aligned} J(m-1, N) &= J_o(m, N) + F(m-1) \\ &= [\mathbf{A}(m-1)\mathbf{x}(m-1) + \mathbf{B}(m-1)\mathbf{u}(m-1)]^T\mathbf{M}(m)[\mathbf{A}(m-1)\mathbf{x}(m-1) \\ &\quad + \mathbf{B}(m-1)\mathbf{u}(m-1)] + \mathbf{x}^{\mathbf{T}}(m-1)\mathbf{Q}(m-1)\mathbf{x}(m-1) \\ &\quad + \mathbf{u}^{\mathbf{T}}(m-1)\mathbf{R}(m-1)\mathbf{u}(m-1) \end{aligned} \tag{8.110}$$

Then the minimization of $J(m-1, N)$ with respect to $\mathbf{u}(m-1)$ yields

$$\partial J(m-1, N)/\partial\mathbf{u}(m-1) = 2\mathbf{B}^{\mathbf{T}}(m-1)\mathbf{M}(m)[\mathbf{A}(m-1)\mathbf{x}(m-1) + \mathbf{B}(m-1)\mathbf{u}(m-1)] \\ + 2\mathbf{R}(m-1)\mathbf{u}(m-1) = \mathbf{0} \tag{8.111}$$

or

$$\mathbf{u_o}(m-1) = -[\mathbf{B}^{\mathbf{T}}(m-1)\mathbf{M}(m)\mathbf{B}(m-1) + \mathbf{R}(m-1)]^{-1}\mathbf{B}^{\mathbf{T}}(m-1)\mathbf{M}(m)\mathbf{A}(m-1)\mathbf{x}(m-1) \tag{8.112}$$

Thus, the optimal regulator matrix at the $(m-1)$th sampling instant is

$$\mathbf{K_o}(m-1) = [\mathbf{B}^{\mathbf{T}}(m-1)\mathbf{M}(m)\mathbf{B}(m-1) + \mathbf{R}(m-1)]^{-1}\mathbf{B}^{\mathbf{T}}(m-1)\mathbf{M}(m)\mathbf{A}(m-1) \tag{8.113}$$

Finally, substituting Eq. (8.112) into Eq. (8.110), we get the following expression for $J_o(m-1, N)$:

$$J_o(m-1, N) = \mathbf{x}^{\mathbf{T}}(m-1)[\mathbf{A}_{\mathbf{c}}^{\mathbf{T}}(m-1)\mathbf{M}(m)\mathbf{A_c}(m-1) + \mathbf{Q}(m-1) + \mathbf{K}^{\mathbf{T}}(m-1)\mathbf{R}(m-1)\mathbf{K}(m-1)]\mathbf{x}(m-1) \tag{8.114}$$

where $\mathbf{A_c}(m-1) = \mathbf{A}(m-1) - \mathbf{B}(m-1)\mathbf{K}(m-1)$. Comparing Eqs. (8.107) and (8.114), it is clear that

$$\mathbf{M}(m-1) = \mathbf{A}_{\mathbf{c}}^{\mathbf{T}}(m-1)\mathbf{M}(m)\mathbf{A_c}(m-1) + \mathbf{Q}(m-1) + \mathbf{K}^{\mathbf{T}}(m-1)\mathbf{R}(m-1)\mathbf{K}(m-1) \tag{8.115}$$

Equation (8.115) is a *nonlinear* matrix difference equation, and can be recognized as the digital equivalent of the *matrix Riccati equation*, which must be integrated *backwards* in time, beginning from the *terminal condition* obtained from Eq. (8.97) as $\mathbf{M}(N) = \mathbf{Q}(N)$.

In summary, the digital, linear quadratic optimal regulator problem consists of the *recursive* solution of the following equations:

$$\mathbf{x}(k+1) = [\mathbf{A}(k) - \mathbf{B}(k)\mathbf{K_o}(k)]\mathbf{x}(k) \tag{8.116}$$

$$\mathbf{K_o}(k) = [\mathbf{B}^{\mathbf{T}}(k)\mathbf{M}(k+1)\mathbf{B}(k) + \mathbf{R}(k)]^{-1}\mathbf{B}^{\mathbf{T}}(k)\mathbf{M}(k+1)\mathbf{A}(k) \tag{8.117}$$

$$\mathbf{M}(k) = [\mathbf{A}(k) - \mathbf{B}(k)\mathbf{K_o}(k)]^T\mathbf{M}(k+1)[\mathbf{A}(k) - \mathbf{B}(k)\mathbf{K_o}(k)] + \mathbf{Q}(k) + \mathbf{K}_{\mathbf{o}}^{\mathbf{T}}(k)\mathbf{R}(k)\mathbf{K_o}(k) \tag{8.118}$$

with the *terminal conditions*, $\mathbf{M}(N) = \mathbf{Q}(N)$ and $\mathbf{K_o}(N) = \mathbf{0}$, and given the *initial condition*, $\mathbf{x}(0)$. Since both initial and terminal conditions are specified for solving Eqs. (8.116)–(8.118), these difference equations pose a *two-point boundary value problem*. Note that the minimum value of the objective function, $J(0, N)$, can be obtained from Eq. (8.108) with $m = 0$ as

$$J_o(0, N) = \mathbf{x}^{\mathbf{T}}(0)\mathbf{M}(0)\mathbf{x}(0) \tag{8.119}$$

While it is possible to solve the two-point boundary value problem posed by Eqs. (8.116)–(8.118) using a nonlinear time marching numerical method (such as the *Runge–Kutta* method discussed in Chapter 4), usually we are interested in a steady state solution, where the terminal time is infinite, or $N \to \infty$. In the limit $N \to \infty$, both the objective function, $J(0, N)$, and the optimal gain matrix, $\mathbf{K_o}(N)$, become constants. It must be pointed out that we are not going to solve Eqs. (8.116)–(8.118) for an *infinite* *number* of sampling instants, but assume that $J(0, N)$ and $\mathbf{K_o}(N)$ approximately become constants as N becomes large. Of course, the closed-loop regulated system given by Eq. (8.116) must be *asymptotically stable* for the steady state approximation to be used. The steady state approximation of the linear optimal control problem is especially valid for *time-invariant* systems for which the optimal regulator gain matrix approaches a *constant* value after a few sampling instants. Let us therefore consider a time-invariant system with $\mathbf{A}(k)$ and $\mathbf{B}(k)$ replaced by the constant matrices $\mathbf{A}$ and $\mathbf{B}$, respectively. Also, for the time-invariant system, the state and control weighting matrices, $\mathbf{Q}$ and $\mathbf{R}$, in the objective function are also constant. In the steady state, the matrix $\mathbf{M}(k)$ becomes constant, and we can write $\mathbf{M}(k) = \mathbf{M}(k+1) = \mathbf{M_o}$, where $\mathbf{M_o}$ is a constant matrix. Thus, in the steady state, Eq. (8.118) can be written as

$$\mathbf{M_o} = (\mathbf{A} - \mathbf{BK_o})^T\mathbf{M_o}(\mathbf{A} - \mathbf{BK_o}) + \mathbf{Q} + \mathbf{K_o^T R K_o} \tag{8.120}$$

and Eq. (8.117) becomes

$$\mathbf{K_o} = (\mathbf{B^T M_o B} + \mathbf{R})^{-1}\mathbf{B^T M_o A} \tag{8.121}$$

With the substitution of Eq. (8.121), Eq. (8.120) can be re-written in the following form, which does not contain $\mathbf{K_o}$:

$$\mathbf{0} = \mathbf{M_o} - \mathbf{A^T M_o A} + \mathbf{A^T M_o B}(\mathbf{R} + \mathbf{B^T M_o B})^{-1}\mathbf{B^T M A} - \mathbf{Q} \tag{8.122}$$

Equation (8.122) is the *digital algebraic Riccati equation* – a digital equivalent of Eq. (6.33). The optimal control gain matrix, $\mathbf{K_o}$, is thus obtained from Eq. (8.121) using the solution, $\mathbf{M_o}$, to the algebraic Riccati equation, Eq. (8.122). A set of *sufficient conditions* for the existence of a *unique, positive definite* solution to Eq. (8.122) are as follows:

(a) The state weighting matrix, $\mathbf{Q}$, must be symmetric and *positive semi-definite.*

(b) The control weighting matrix, $\mathbf{R}$, must be symmetric and *positive definite.*

(c) The digital system represented by $\mathbf{A}$ and $\mathbf{B}$ must be *controllable* (or at least, *stabilizable*).

A commonly employed numerical approach to finding the solution to Eq. (8.122) is by defining a *Hamiltonian matrix*, $\mathcal{H}$, as follows:

$$\begin{bmatrix} \mathbf{x}(k+1) \\ \mathbf{p}(k+1) \end{bmatrix} = \mathcal{H} \begin{bmatrix} \mathbf{x}(k) \\ \mathbf{p}(k) \end{bmatrix} \tag{8.123}$$

where $\mathbf{p}(k)$ is called the *costate vector* and $\mathcal{H}$ is the following:

$$\mathcal{H} = \begin{bmatrix} \mathbf{A} + \mathbf{BR^{-1}B^T}(\mathbf{A}^{-1})^T\mathbf{Q} & -\mathbf{BR^{-1}B^T}(\mathbf{A}^{-1})^T \\ -(\mathbf{A}^{-1})^T\mathbf{Q} & (\mathbf{A}^{-1})^T \end{bmatrix} \tag{8.124}$$

Of course, the definition of the Hamiltonian matrix by Eq. (8.124) requires that $\mathbf{A}$ must be *non-singular.* The Hamiltonian matrix and the costate vector are creatures residing in an *alternative formulation* of the linear optimal control problem, called the *minimum principle* [3], which requires minimizing the Hamiltonian, $\mathcal{H} = 1/2[\mathbf{x}^T(k)\mathbf{Q}\mathbf{x}(k) + \mathbf{u}^T(k)\mathbf{R}\mathbf{u}(k)] + \mathbf{p}^T(k+1)[\mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)]$ with respect to the control input, $\mathbf{u}(k)$. Note that $\mathcal{H}$ includes the term $\mathbf{p}^T(k+1)[\mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)]$ as a *penalty* for deviating from the system's state-equation, $\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$, which was a constraint in the minimization of the objective function, $J(0, N)$. The optimal control input resulting from the minimization of $\mathcal{H}$ is exactly the same as that obtained earlier in this section. In terms of the costate vector, $\mathbf{p}(k)$, the optimal control input can be expressed as follows [3]:

$$\mathbf{u}_o(k) = -\mathbf{R}^{-1}\mathbf{B}^T\mathbf{p}(k+1) \tag{8.125}$$

Now, since Eq. (8.125) requires marching *backwards* in time, it would be mores useful to express Eq. (8.123) as follows:

$$\begin{bmatrix} \mathbf{x}(k) \\ \mathbf{p}(k) \end{bmatrix} = \mathcal{H}^{-1} \begin{bmatrix} \mathbf{x}(k+1) \\ \mathbf{p}(k+1) \end{bmatrix} \tag{8.126}$$

It can be shown easily that the inverse of the Hamiltonian matrix is given by

$$\mathcal{H}^{-1} = \begin{bmatrix} \mathbf{A}^{-1} & \mathbf{A}^{-1}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T \\ \mathbf{Q}\mathbf{A}^{-1} & \mathbf{A}^T + \mathbf{Q}\mathbf{A}^{-1}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T \end{bmatrix} \tag{8.127}$$

The Hamiltonian matrix has an interesting property that if λ is an eigenvalue of $\mathcal{H}$, then $1/\lambda$ is an eigenvalue of $\mathcal{H}^{-1}$. Also, the eigenvalues of $\mathcal{H}$ are the eigenvalues of $\mathcal{H}^{-1}$. Hence, it follows that the eigenvalues of $\mathcal{H}$ (or $\mathcal{H}^{-1}$) must occur in *reciprocal pairs*, i.e. λ_1, $1/\lambda_1$, λ_2, $1/\lambda_2$, etc. If the eigenvalues of $\mathcal{H}$ (or $\mathcal{H}^{-1}$) are *distinct*, then we can *diagonalize* the state-equations, (8.126) (see Chapter 3). The state-equations given by Eq. (8.126) can be diagonalized by a transformation matrix, $\mathbf{T} = \mathbf{V}^{-1}$, where $\mathbf{V}$ is a *modal matrix* whose columns are the eigenvectors of $\mathcal{H}^{-1}$. If we partition $\mathbf{V}$ into *four* $(n \times n)$ sized blocks (where n is the order of the system) as follows:

$$\mathbf{V} = \begin{bmatrix} \mathbf{V}_{11} & \mathbf{V}_{12} \\ \mathbf{V}_{21} & \mathbf{V}_{22} \end{bmatrix} \tag{8.128}$$

such that

$$\mathbf{D} = \mathbf{V}^{-1}\mathcal{H}^{-1}\mathbf{V} = \begin{bmatrix} \Lambda & \mathbf{0} \\ \mathbf{0} & \Lambda^{-1} \end{bmatrix} \tag{8.129}$$

where Λ is the *diagonal* matrix consisting of the eigenvalues of $\mathcal{H}$ (or $\mathcal{H}^{-1}$) that lie *inside* the unit circle in the z-plane. Clearly, Λ^{-1} is the *diagonal* matrix consisting of the eigenvalues of $\mathcal{H}$ (or $\mathcal{H}^{-1}$) that lie *outside* the unit circle in the z-plane. Equation (8.129) suggests that the modal matrix, $\mathbf{V}$, is partitioned into *stable* and *unstable* eigenvalues of $\mathcal{H}$ (or $\mathcal{H}^{-1}$). Comparing Eqs. (8.125) and (8.112) in the limit $k \to \infty$ it can be shown [2] that

$$\mathbf{M}_o = \mathbf{V}_{21}\mathbf{V}_{11}^{-1} \tag{8.130}$$

Equation (8.130) gives us a direct method of calculating the solution to the algebraic Riccati equation, $\mathbf{M}_o$. However, use of Eq. (8.130) imposes an *additional* condition to the sufficient conditions for the existence of a unique positive definite solution of the algebraic Riccati equation, namely that **A** must be *non-singular*. We can use Eq. (8.130) by constructing the Hamiltonian matrix, finding the eigenvectors of its inverse, and partitioning the modal matrix, **V**, according to Eq. (8.128), or simply use the specialized MATLAB (CST) command *dlqr* which does the same thing. The command *dlqr* is the digital equivalent of the command *lqr* for solving the analog algebraic Riccati equation, and is used as follows:

```
>>[Ko,Mo,E] = dlqr(A,B,Q,R) <enter>
```

where the matrices **A, B, Q, R**, $\mathbf{K_o}$, and $\mathbf{M_o}$ are the same as in the foregoing discussion, while **E** is the returned matrix of the eigenvalues of $\mathbf{A - BK_o}$.

Example 8.20

Let us design an optimal regulator for a digital turbo-generator with the following *digital*, linear, time-invariant state-space representation:

$$\mathbf{A} = \begin{bmatrix} 0.1346 & 0.1236 & -0.0361 & 0.0037 & 0.0004 & -0.0003 \\ -0.1091 & 0.5412 & 0.3851 & -0.0631 & -0.0520 & 0.0152 \\ 0.0426 & 0.1052 & 0.7915 & 0.0700 & 0.0504 & -0.0172 \\ -0.0045 & 0.0205 & -0.0542 & 0.7932 & -0.5687 & 0.0025 \\ 0.0022 & -0.0150 & 0.0384 & 0.5681 & 0.7526 & 0.0357 \\ 0.0002 & -0.0162 & 0.0142 & 0.0004 & -0.0299 & 0.9784 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} -0.0121 & 0.0117 \\ -0.0046 & -0.4960 \\ -0.0150 & 0.5517 \\ 0.0095 & -0.1763 \\ -0.0055 & -1.0216 \\ 0.0025 & 1.3944 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 0.5971 & -0.7697 & 4.8850 & 4.8608 & -9.8177 & -8.8610 \\ 3.1013 & 9.3422 & -5.6000 & -0.7490 & 2.9974 & 10.5719 \end{bmatrix} \tag{8.131}$$

$$\mathbf{D} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The two inputs are the *throttle-valve position*, $u_1(k)$, and the *loading torque*, $u_2(k)$, while the two outputs are the *deviation from the desired generated voltage* (or *voltage error*), $y_1(k)$ (volt), and the *deviation of the generator load's angular position*, (or *load position error*), $y_2(k)$ (radians). For this sixth-order system, it is desired that the closed-loop initial response to the initial condition, $\mathbf{x}(0) = [0.1; 0; 0; 0; 0; 0]^T$ should decay to zero in about 20 sampling instants, with a maximum overshoot in $y_1(k)$ of 0.1 V and in $y_2(k)$ of 0.35 rad. For the plant, the first output takes about 50 samples to settle to zero. We select the weighting matrices for the optimal control problem as $\mathbf{Q} = \mathbf{I}$ and $\mathbf{R} = \mathbf{I}$. The inverse of the Hamiltonian matrix, $\mathcal{H}^{-1}$, and its eigenvalues are calculated as follows:

```
>>Q=eye(6); R=eye(2); <enter>

>>Ainv=inv(A); Rp=B*inv(R)*B'; Hinv = [Ainv Ainv*Rp; Q*Ainv A'
  +Q*Ainv*Rp] <enter>

Hinv =

Columns 1 through 7
 5.8602   -1.5384    1.0209   -0.0739   -0.2316    0.0523    0.0211
 1.5564    1.6366   -0.7308    0.0475    0.1953   -0.0451   -0.0170
-0.5237   -0.1360    1.3087   -0.0359   -0.1227    0.0295    0.0114
-0.0100   -0.0193    0.0356    0.8172    0.6128   -0.0236   -0.0091
 0.0461    0.0572   -0.1095   -0.6130    0.8754   -0.0332   -0.0107
 0.0338    0.0311   -0.0347   -0.0178    0.0316    1.0199    0.0159
 5.8602   -1.5384    1.0209   -0.0739   -0.2316    0.0523    0.1557
 1.5564    1.6366   -0.7308    0.0475    0.1953   -0.0451    0.1065
-0.5237   -0.1360    1.3087   -0.0359   -0.1227    0.0295   -0.0247
-0.0100   -0.0193    0.0356    0.8172    0.6128   -0.0236   -0.0054
 0.0461    0.0572   -0.1095   -0.6130    0.8754   -0.0332   -0.0103
 0.0338    0.0311   -0.0347   -0.0178    0.0316    1.0199    0.0156

Columns 8 through 12
-0.8515    0.9488   -0.3035   -1.7544    2.3950
 0.7278   -0.8093    0.2585    1.4991   -2.0461
-0.4742    0.5277   -0.1687   -0.9768    1.3333
 0.3838   -0.4270    0.1365    0.7906   -1.0792
 0.4567   -0.5079    0.1622    0.9408   -1.2840
-0.6740    0.7497   -0.2395   -1.3883    1.8950
-0.9606    0.9914   -0.3080   -1.7521    2.3951
 1.2691   -0.7041    0.2790    1.4842   -2.0623
-0.0890    1.3193   -0.2229   -0.9384    1.3476
 0.3207   -0.3571    0.9297    1.3588   -1.0787
 0.4047   -0.4574   -0.4064    1.6934   -1.3139
-0.6588    0.7325   -0.2370   -1.3526    2.8733

>>eig(Hinv) <enter>
ans =
  5.7163 + 1.8038i
  5.7163 - 1.8038i
  2.0510
  1.0596 + 0.6776i
  1.0596 - 0.6776i
  0.1591 + 0.0502i
  0.1591 - 0.0502i
  0.6698 + 0.4283i
  0.6698 - 0.4283i
  0.4876
  1.1062
  0.9040
```

Note that the eigenvalues of $\mathcal{H}^{-1}$ occur in reciprocal stable and unstable pairs, as expected. Then, the steady state solution to the algebraic Riccati equation, $\mathbf{M_o}$, the optimal regulator gain matrix, $\mathbf{K_o}$, and the set of closed-loop eigenvalues, $\mathbf{E}$, can be obtained using *dlqr* as follows:

```
>>[Ko, Mo, E] = dlqr(A, B, Q, R) <enter>

Ko =
 -0.0027  -0.0267  -0.0886   0.0115  -0.0375   0.0216
  0.0024  -0.0025   0.0223   0.0514  -0.4316   0.3549

Mo =
  1.0375  -0.0491   0.0001   0.0495   0.0678  -0.0246
 -0.0491   1.8085   1.6151   0.1214  -0.2535  -0.3059
  0.0001   1.6151   5.8619   0.7194   0.3663  -1.0029
  0.0495   0.1214   0.7194   5.2730  -0.5666   1.6496
  0.0678  -0.2535   0.3663  -0.5666   4.1756   0.8942
 -0.0246  -0.3059  -1.0029   1.6496   0.8942   3.2129

E =
 0.1591 + 0.0502i
 0.1591 - 0.0502i
 0.4876
 0.6698 + 0.4283i
 0.6698 - 0.4283i
 0.9040
```

The closed-loop initial response is obtained using *dinitial* as follows:

```
>>dinitial(A-B*Ko,B,C,D,[0.1 zeros(1,5)]') <enter>
```

The resulting outputs, $y_1(k)$ and $y_2(k)$, are plotted in Figure 8.13. Note that the requirement of both outputs settling to zero in about 20 sampling instants, with specified maximum overshoots, has been met.

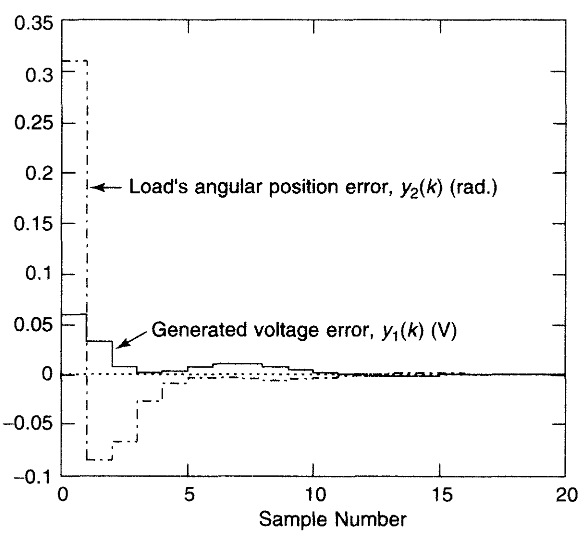

Figure 8.13 Initial response of the optimally regulated digital turbo-generator system