

As in Chapter 6, we can formulate the digital equivalent of the *output weighted* linear, optimal control, where the weightage is placed on the output rather than on the state-vector. The objective function for such a problem for the time-invariant case can be written as

$$J(0, N) = \sum_{k=0}^N \mathbf{y}^T(k) \mathbf{Q} \mathbf{y}(k) + \mathbf{u}^T(k) \mathbf{R} \mathbf{u}(k) \quad (8.132)$$

which, by the substitution of the output equation, $\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{D}\mathbf{u}(k)$, becomes of the following form:

$$J(0, N) = \sum_{k=0}^N \mathbf{x}^T(k) \mathbf{Q}^* \mathbf{x}(k) + 2\mathbf{u}^T(k) \mathbf{N} \mathbf{x}(k) + \mathbf{u}^T(k) \mathbf{R} \mathbf{u}(k) \quad (8.133)$$

Equation (8.133) is the general quadratic form of the quadratic objective function, and can be minimized using *dlqr* with a fourth additional input argument, the cross-weighting matrix, $\mathbf{N}$. You can derive the algebraic Riccati equation for the steady state solution of the output weighted problem along the lines of the foregoing discussion. The MATLAB (CST) directly provides the solution for the output weighted, steady state, linear, time-invariant, optimal control problem through its command *dlqry* which is used as follows:

```
>>[Ko,Mo,E] = dlqry(A,B,C,D,Q,R) <enter>
```

where the symbols have their usual meanings.

The solution to the digital terminal-time weighted linear optimal control problem for a time-invariant plant can be obtained along the lines of Section 6.5, with suitable modifications. You are encouraged to write a computer program – a digital equivalent of *tpbvlti.m* – for solving the two-point boundary value problem for the terminal-time weighted digital optimal control.

8.12 Stochastic Digital Systems, Digital Kalman Filters, and Optimal Digital Compensators

There is no fundamental difference in the way random (or stochastic) variables are handled in modeling and control of digital and analog systems. Hence, the methods of Chapter 7 can be applied on a one-to-one basis to stochastic systems, of course with the understanding that the s -plane is transformed to the z -plane, and an analog state-space representation is transformed into a digital representation. However, there is a fundamental difference in the manner in which digital and analog systems respond to noise. As we have seen earlier in this chapter, a digital control system's frequency response is limited to signals whose frequencies are *less than* the Nyquist frequency, π/T . Hence, a digital control system is inherently *robust* with respect to high-frequency noise. This is precisely why digital control systems have become so popular in applications where high-frequency noise is present. For example, a digital video disc (DVD) player shows

much clearer pictures than the analog video cassette player (VCP), primarily due to this inherent robustness.

Since any noise with frequency higher than the Nyquist frequency gets automatically cut-off by a digital control system, we have to only worry about noise with frequency contents smaller than the Nyquist frequency. In a manner similar to analog filters, we can define a *digital filter* as any digital system through which a noise is input. For single-input, single-output applications, a digital filter has a pulse transfer function, $G(z)$. Depending on the application, we can design *low-pass*, *band-pass*, or *high-pass* digital filters with appropriate pulse transfer functions using an extension of the techniques of Chapter 7.

Example 8.21

Let us construct a first order low-pass digital filter to cut-off a uniformly distributed random noise superimposed on a signal, $\sin(10t)$. The pulse transfer function of the filter is given by $G(z) = a/(z + b)$, where a and b are constants. Let us also initially select a sampling interval, $T = 0.1$ s. The Nyquist frequency is $\pi/T = 31.4$ rad/s. Hence, the noise with frequencies higher than 31.4 rad/s gets automatically cut-off. Since the input is a continuous time signal, the filter is selected to be a sampled-data analog filter with transfer function, $G(s) = 15/(s + 15)$ and a zero-order hold. Then $G(z)$ is obtained using *c2dm* as follows:

```
>>num = 15; den = [1 15]; [numd, dend] = c2dm(num,den,0.1,'zoh')
<enter>

numd2 =
    0      0.7769

dend2 =
    1.0000   -0.2231
```

Hence, $G(z) = 0.7769/(z - 0.2231)$. The response of this filter to the noisy signal can be simulated with the following command:

```
>>t=0:0.05:1; u = sin(10*t)+randn(size(t));
y = dlsim(numd,dend,u); <enter>
```

The most important extension of the analog stochastic concepts of Chapter 7 for digital stochastic systems is the *digital Kalman filter*. The digital Kalman filter is an *optimal digital observer* for a noisy plant. Consider a linear, time varying plant with the following digital state-space representation:

$$\mathbf{x}(k+1) = \mathbf{A}(k)\mathbf{x}(k) + \mathbf{B}(k)\mathbf{u}(k) + \mathbf{F}(k)\mathbf{v}(k) \quad (8.134)$$

$$\mathbf{y}(k) = \mathbf{C}(k)\mathbf{x}(k) + \mathbf{D}(k)\mathbf{u}(k) + \mathbf{z}(k) \quad (8.135)$$

where $\mathbf{v}(k)$ is the *process noise vector*, which may arise due to modeling errors such as neglecting nonlinear or higher-frequency dynamics, and $\mathbf{z}(k)$ is the *measurement noise vector*. By assuming $\mathbf{v}(k)$ and $\mathbf{z}(k)$ to be discrete, *uncorrelated white noises*, we will only

be extending the methodology of Chapter 7 for a description of the digital stochastic plant. The *correlation matrices* of *non-stationary* discrete white noises, $\mathbf{v}(k)$ and $\mathbf{z}(k)$, can be expressed as follows:

$$\mathbf{R}_v(j, k) = \mathbf{V}(k)\delta_{jk} \quad (8.136)$$

$$\mathbf{R}_z(j, k) = \mathbf{Z}(k)\delta_{jk} \quad (8.137)$$

where $\mathbf{V}(k)$ and $\mathbf{Z}(k)$ are the power spectral density matrices (also the covariance matrices) of $\mathbf{v}(k)$ and $\mathbf{z}(k)$, respectively, and δ_{jk} is the *Kronecker delta function* ($\delta_{jk} = 0$ when $j \neq k$, and $\delta_{jk} = 1$ when $j = k$). The digital, time-varying Kalman filter's state-equation is written as that of the *current observer* (Eq. (8.86)) in the following manner:

$$\begin{aligned} \mathbf{x}_o(k+1) = & [\mathbf{A}(k) - \mathbf{L}^o(k)\mathbf{C}(k)\mathbf{A}(k)]\mathbf{x}_o(k) + [\mathbf{B}(k) - \mathbf{L}^o(k)\mathbf{C}(k)\mathbf{B}(k)]\mathbf{u}(k) \\ & - \mathbf{L}^o(k)\mathbf{D}(k)\mathbf{u}(k+1) + \mathbf{L}^o(k)\mathbf{y}(k+1) \end{aligned} \quad (8.138)$$

where the optimal gain matrix, $\mathbf{L}^o(k)$, is to be determined by *minimizing* the *covariance* of the estimation error vector, $\mathbf{e}_o(k)$. Since the estimation error vector, $\mathbf{e}_o(k)$, is a discrete random variable, we can use the methods of Section 7.4 for obtaining a corresponding difference equation to be satisfied by the optimal covariance matrix. However, recall that in Section 7.4, we had only minimized a *conditional covariance matrix* of estimation error (i.e. the covariance matrix based on a *finite record* of the output). The digital equivalent of the conditional covariance matrix is the covariance matrix of the *prediction error vector*, $\mathbf{e}_1(k+1) = [\mathbf{x}(k+1) - \mathbf{x}_1(k+1)]$, which is based on the measurement of the output at the *previous* sampling instant, $\mathbf{y}(k)$ (see Eq. (8.87)). Recall that the current observer state-equation can be alternatively expressed as two *predictor-corrector* equations, Eqs. (8.87) and (8.88). For the optimal current observer given by Eq. (8.138), the predictor-corrector difference equations are the following:

$$\mathbf{x}_1(k+1) = \mathbf{A}(k)\mathbf{x}_o(k) + \mathbf{B}(k)\mathbf{u}(k) \quad (8.139)$$

$$\begin{aligned} \mathbf{x}_o(k+1) = & \mathbf{x}_1(k+1) + \mathbf{L}^o(k+1)[\mathbf{y}(k+1) - \mathbf{D}(k+1)\mathbf{u}(k+1) \\ & - \mathbf{C}(k+1)\mathbf{x}_1(k+1)] \end{aligned} \quad (8.140)$$

where $\mathbf{x}_1(k+1)$ is the *predicted state-vector* at the $(k+1)$ th sampling instant based upon the quantities at the *previous* sampling instant, and $\mathbf{x}_o(k+1)$ is the *corrected state-vector* estimate due to the measurement of the output at the current sampling instant, $\mathbf{y}(k+1)$. Defining $\mathbf{R}_e^o(k)$ as the *optimal covariance matrix* of the *prediction error*, we can write

$$\mathbf{R}_e^o(k) = \mathbf{E}[\mathbf{e}_1(k)\mathbf{e}_1^T(k)] = \mathbf{E}\{[\mathbf{x}(k) - \mathbf{x}_1(k)]\{[\mathbf{x}(k) - \mathbf{x}_1(k)]^T\} \quad (8.141)$$

whereas the optimal covariance matrix of the *corrected* (or *true*) estimation error is written as

$$\mathbf{P}^o(k) = \mathbf{E}[\mathbf{e}_o(k)\mathbf{e}_o^T(k)] = \mathbf{E}\{[\mathbf{x}(k) - \mathbf{x}_o(k)]\{[\mathbf{x}(k) - \mathbf{x}_o(k)]^T\} \quad (8.142)$$

You can show, using these definitions and the method of Section 7.4, that for the linear, time-varying, digital Kalman filter, the following difference equations must be satisfied

by the optimal covariance matrix, $\mathbf{R}_e^0(k)$:

$$\mathbf{R}_e^0(k+1) = \mathbf{A}(k)[\mathbf{I} - \mathbf{L}^0(k)\mathbf{C}(k)]\mathbf{R}_e^0(k)\mathbf{A}^T(k) + \mathbf{F}(k)\mathbf{V}(k)\mathbf{F}^T(k) \quad (8.143)$$

where the Kalman filter gain, $\mathbf{L}^0(k)$, is given by

$$\mathbf{L}^0(k) = \mathbf{R}_e^0(k)\mathbf{C}^T(k)[\mathbf{C}(k)\mathbf{R}_e^0(k)\mathbf{C}^T(k) + \mathbf{Z}(k)]^{-1} \quad (8.144)$$

For a linear time invariant plant in the steady state, the state coefficient matrices, the power spectral densities of the noise processes, the Kalman filter gain, and the optimal covariance matrix, all become constants. In such a case, Eq. (8.143) becomes

$$\mathbf{R}_e^0 = \mathbf{A}(\mathbf{I} - \mathbf{L}^0\mathbf{C})\mathbf{R}_e^0\mathbf{A}^T + \mathbf{F}\mathbf{V}\mathbf{F}^T \quad (8.145)$$

where

$$\mathbf{L}^0 = \mathbf{R}_e^0\mathbf{C}^T(\mathbf{C}\mathbf{R}_e^0\mathbf{C}^T + \mathbf{Z})^{-1} \quad (8.146)$$

Substituting Eq. (8.146) into Eq. (8.145) we get the following explicit equation in terms of the unknown matrix, $\mathbf{R}_e^0$:

$$\mathbf{R}_e^0 = \mathbf{A}\mathbf{R}_e^0\mathbf{A}^T - \mathbf{A}\mathbf{R}_e^0\mathbf{C}^T(\mathbf{C}\mathbf{R}_e^0\mathbf{C}^T + \mathbf{Z})^{-1}\mathbf{C}\mathbf{R}_e^0\mathbf{A}^T + \mathbf{F}\mathbf{V}\mathbf{F}^T \quad (8.147)$$

Equations (8.146) and (8.147) should be compared with Eqs. (8.121) and (8.122), respectively, for the digital optimal regulator problem. It can be seen that the two sets of equations are *identical* in form, with the corresponding matrices compared in Table 8.2. Due to this similarity, the Kalman filter and optimal regulator are called *dual* problems. An alternative form of Eqs. (8.146) and (8.147) in terms of the optimal covariance matrix of the estimation error, $\mathbf{P}^0$, can be written as follows:

$$\mathbf{L}^0 = \mathbf{P}^0\mathbf{C}^T\mathbf{Z}^{-1} \quad (8.148)$$

$$\mathbf{R}_e^0 = \mathbf{A}\mathbf{P}^0\mathbf{A}^T + \mathbf{F}\mathbf{V}\mathbf{F}^T \quad (8.149)$$

where

$$\mathbf{P}^0 = \mathbf{R}_e^0 - \mathbf{R}_e^0\mathbf{C}^T(\mathbf{C}\mathbf{R}_e^0\mathbf{C}^T + \mathbf{Z})^{-1}\mathbf{C}\mathbf{R}_e^0 \quad (8.150)$$

From Table 8.2 it is clear that Eq. (8.147) is none other than the *algebraic Riccati equation*, which can be solved by the techniques of the previous section. After the solution,

Table 8.2 Comparison of analogous matrices for the digital Kalman filter and the digital optimal regulator

Digital Optimal Regulator	Digital Kalman Filter
$\mathbf{A}$	$\mathbf{A}^T$
$\mathbf{B}$	$\mathbf{C}^T$
$\mathbf{M}_0$	$\mathbf{R}_e^0$
$\mathbf{Q}$	$\mathbf{F}\mathbf{V}\mathbf{F}^T$
$\mathbf{R}$	$\mathbf{Z}$
$\mathbf{K}$	$\mathbf{P}^0 = \mathbf{R}_e^0 - \mathbf{R}_e^0\mathbf{C}^T(\mathbf{C}\mathbf{R}_e^0\mathbf{C}^T + \mathbf{Z})^{-1}\mathbf{C}\mathbf{R}_e^0$

$\mathbf{R}_e^o$, of the algebraic Riccati equation is obtained, it can be substituted into Eq. (8.146) to get the Kalman filter gain matrix, $\mathbf{L}^o$.

The inverse of the Hamiltonian matrix for the steady state Kalman filter is given by

$$\mathcal{H}^{-1} = \begin{bmatrix} (\mathbf{A}^T)^{-1} & (\mathbf{A}^T)^{-1} \mathbf{C}^T \mathbf{Z}^{-1} \mathbf{C} \\ \mathbf{F} \mathbf{V} \mathbf{F}^T (\mathbf{A}^T)^{-1} & \mathbf{A} + \mathbf{F} \mathbf{V} \mathbf{F}^T (\mathbf{A}^T)^{-1} \mathbf{C}^T \mathbf{Z}^{-1} \mathbf{C} \end{bmatrix} \quad (8.151)$$

Then the steps given by Eqs. (8.127)–(8.130) can be extended for obtaining the solution to the algebraic Riccati equation. We can either use the MATLAB (CST) command *dlqr* with the appropriate substitution of the regulator matrices given by Table 8.2 for obtaining the Kalman filter gain, or the specialized CST command *dlqe*, which directly uses the Kalman filter matrices. The command *dlqe* is used as follows:

```
>>[Lo,Ro,Po,E] = dlqe(A,F,C,V,Z) <enter>
```

where $\mathbf{A}$, $\mathbf{F}$, and $\mathbf{C}$ are the state coefficient matrices of the plant, $\mathbf{V}$ and $\mathbf{Z}$ are the covariance matrices of the process and measurement noise, and $\mathbf{L}^o = \mathbf{L}^o$, the returned Kalman filter gain, $\mathbf{R}^o = \mathbf{R}_e^o$, the returned optimal covariance matrix of the predicted estimation error, $\mathbf{P}^o = \mathbf{P}^o$, the returned optimal covariance matrix of the estimation error, and $\mathbf{E}$ is a vector containing the eigenvalues of the digital Kalman filter (i.e. the eigenvalues of $\mathbf{A} - \mathbf{L}^o \mathbf{C} \mathbf{A}$). When used with a *fifth* input argument, $\mathbf{N}$, *dlqe* solves the Kalman filter gain when the process and measurement noises are *correlated*, with the cross-covariance matrix, $\mathbf{N} = \mathbf{E}[\mathbf{v}(k)\mathbf{z}^T(k)]$. MATLAB also provides the command *dlqew* to solve for the Kalman filter gain when the process noise directly affects the output, and the process and measurement noises are uncorrelated.

An optimal digital compensator can be formed with the optimal regulator and the Kalman filter using the methodology of Section 8.10. Such a compensator is called the *digital, linear quadratic gaussian* (DLQG) compensator, and is the digital counterpart of the LQG compensator of Chapter 7.

Example 8.22

Let us derive a Kalman filter for the turbo-generator of Example 8.20, and combine the optimal regulator and the Kalman filter into an optimal digital compensator. Assuming that the process noise, $\mathbf{v}(k)$, consists of two variables, with the coefficient matrix, $\mathbf{F} = \mathbf{B}$, and the covariance matrix, $\mathbf{V} = \mathbf{I}$. The measurement noise is assumed to have a covariance matrix, $\mathbf{Z} = \mathbf{I}$. The Kalman filter gain and the optimal covariance matrices of the prediction and estimation errors are obtained using *dlqe* as follows:

```
>>[Lo, Ro, Po, E]=dlqe(A, B, C, eye(2), eye(2)) <enter>
```

```
Lo =
-0.0011 -0.0037
-0.0041 -0.0666
0.0112 0.0881
-0.0054 -0.0118
-0.0503 -0.1789
0.0219 0.2481
```

$R_o =$

0.0025	-0.0044	0.0000	-0.0058	0.0066	-0.0102
-0.0044	0.2479	-0.2770	0.0864	0.5165	-0.7057
0.0000	-0.2770	0.3263	-0.0859	-0.6216	0.8529
-0.0058	0.0864	-0.0859	0.0448	0.1470	-0.1939
0.0066	0.5165	-0.6216	0.1470	1.2086	-1.6557
-0.0102	-0.7057	0.8529	-0.1939	-1.6557	2.2813

$P_o =$

0.0020	-0.0122	0.0102	-0.0071	-0.0142	0.0187
-0.0122	0.1067	-0.0905	0.0617	0.1393	-0.1801
0.0102	-0.0905	0.0800	-0.0532	-0.1233	0.1589
-0.0071	0.0617	-0.0532	0.0404	0.0806	-0.1017
-0.0142	0.1393	-0.1233	0.0806	0.1994	-0.2521
0.0187	-0.1801	0.1589	-0.1017	-0.2521	0.3255

$E =$

0.0577
 -0.0957
 -0.3994
 0.6125 + 0.5153i
 0.6125 - 0.5153i
 0.8824

All the Kalman filter eigenvalues are seen to be inside the unit circle. An optimal regulator for this plant was designed in Example 8.20 with the following regulator gain matrix:

$$K_o = \begin{bmatrix} -0.0027 & -0.0267 & -0.0886 & 0.0115 & -0.0375 & 0.0216 \\ 0.0024 & -0.0025 & 0.0223 & 0.0514 & -0.4316 & 0.3549 \end{bmatrix}$$

We can use this regulator matrix and the Kalman filter gain calculated above to form a digital optimal compensator as follows:

```
>>[ac,bc,cc,dc] = dreg(A,B,C,D,Ko,Lo)%  
digital compensator's state-space model <enter>
```

ac =

0.1788	0.2529	-0.1088	-0.0003	0.0329	0.1310
-0.4141	-0.3308	0.8289	-0.0642	-0.3533	-0.6335
0.1827	0.5037	0.5889	0.0634	0.3088	0.1481
-0.3818	-1.0436	0.4609	0.7491	-0.7100	-0.9088
-0.1208	-0.4202	0.3444	0.6989	0.0907	-0.1336
-0.0477	-0.1929	0.1706	0.0292	0.3426	0.1362

bc =

-0.0014	-0.0140
0.0201	0.0949
-0.0111	-0.0435
0.0288	0.1163
-0.0096	0.0423
-0.0179	0.0178

```

cc =
    -0.0219    -0.0809    -0.0624     0.0091    -0.0413    -0.0284
    -0.5325    -1.5382     0.8126     0.0332    -0.6425    -1.1477

dc =
     0.0014     0.0059
     0.0294     0.1668

>>sysc = ss(ac,bc,cc,dc); sysp = ss(A,B,C,D); <enter>

>>syss = series(sysc,sysp);
    sysCL = feedback(syss,eye(size(syss))); <enter>

>>[aCL, bCL, cCL, dCL] = ssdata(sysCL); %
    closed-loop system's state-space model <enter>

```

The closed-loop initial response (with the same initial condition as in Example 8.20) is now calculated and plotted in Figure 8.14 using the following command:

```
>>dinitial(aCL,bCL,cCL,dCL,[0.1 zeros(1,11)]') <enter>
```

Comparing Figures 8.13 and 8.14, we find that the DLQG compensated system has a slightly deteriorated performance when compared to the full-state feedback regulated system of Example 8.20 (the overshoots and steady-state error are larger).

Finally, we test the robustness of the DLQG compensated system to white noise disturbances occurring at the compensator's input. For this purpose, we construct a SIMULINK block diagram shown in Figure 8.15, consisting of the plant connected in closed-loop with the DLQG compensator. A two channel simultaneous white

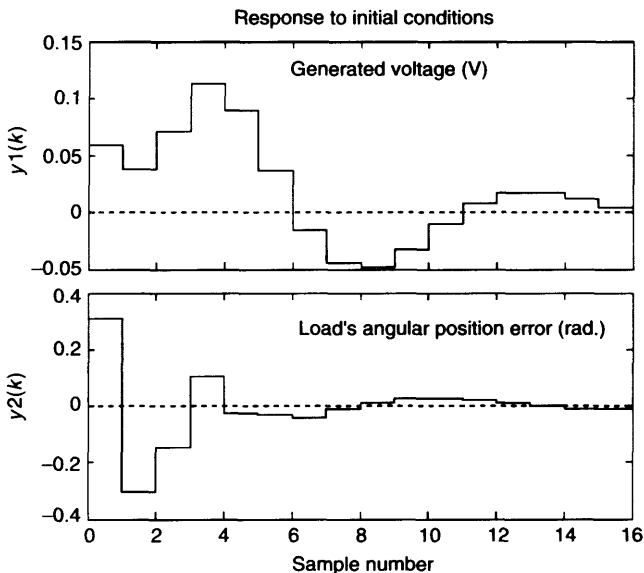


Figure 8.14 Initial response of the DLQG compensated digital turbo-generator system

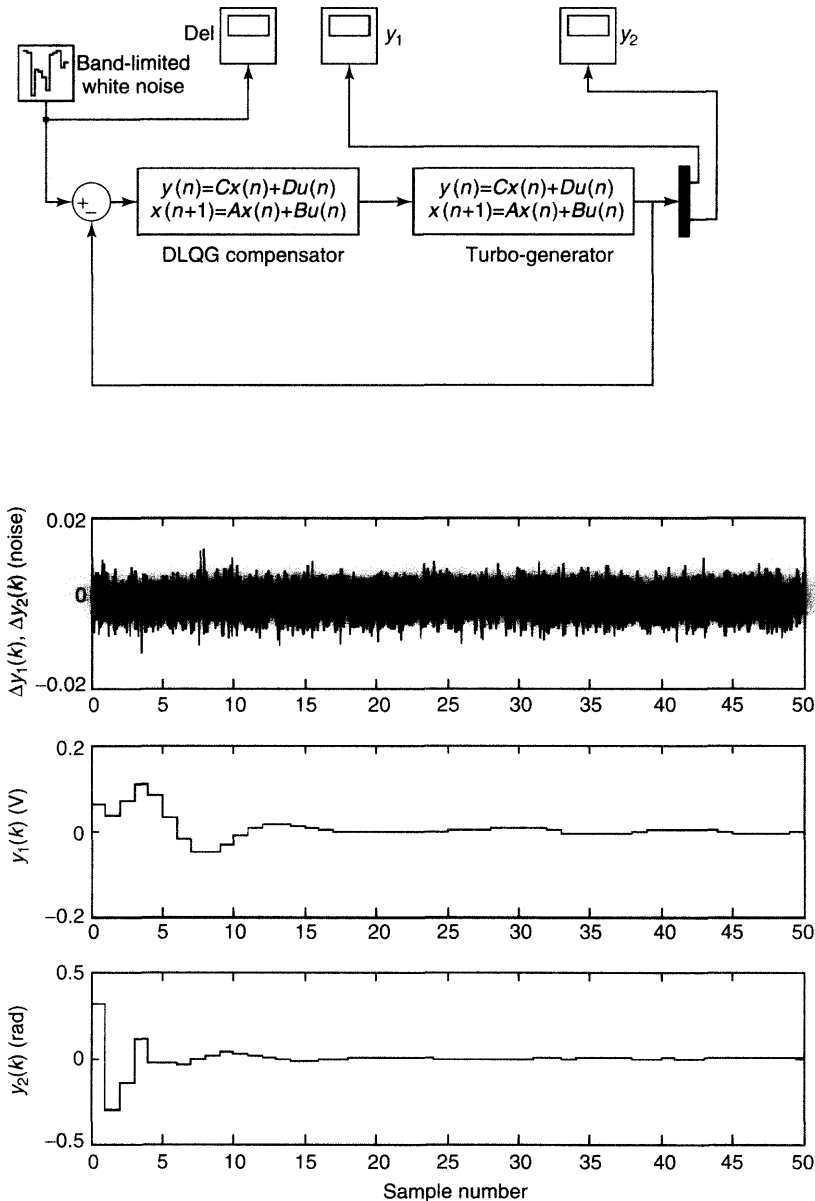


Figure 8.15 Simulation of the DLQG compensated turbo-generator with simultaneous white noise disturbances in each output channel

noise input to the control system is provided through the *band-limited white noise* block, with the noise power set at 10^{-6} at a sample interval of 0.01 s. The simulated initial response of $y_1(k)$ and $y_2(k)$ is seen in Figure 8.15 to be quite unaffected by the applied disturbances, which are of very high frequency. This simulation illustrates the built-in robustness of digital systems to high-frequency noise.