(in Hertz) of the analog signal. This minimum value of the sampling rate is known as the *Nyquist sampling rate*.

It is clear from the above discussion that a successful implementation of a digital control system requires a *mathematical model* for the A/D converter. Since A/D converter is based on *two* distinct processes – *sampling* and *holding* – such a model must include separate models for both of these processes. As with continuous time systems, we can obtain the mathematical model for A/D converter in either *frequency domain* (using transform methods and transfer function), or in the *time domain* (using a state-space representation). We begin with the frequency domain modeling and analysis of single-input, single-output digital systems.

8.2 A/D Conversion and the z-Transform

The simplest model for the *sampling process* of the A/D converter is a *switch* which repeatedly closes for a *very short duration*, t_w, after every T seconds, where $1/T$ is the *sampling rate* of the analog input, $f(t)$. The output of such a switch would consist of *series* of *pulses* separated by T seconds. The *width* of each pulse is the duration, t_w, for which the switch remains closed. If t_w is very small in comparison to T, we can assume that $f(t)$ *remains constant* during each pulse (i.e. the pulses are rectangular). Thus, the *height* of the kth pulse is the value of the analog input $f(t)$ at $t = kT$, i.e. $f(kT)$. If t_w is very small, the kth pulse can be approximated by a *unit impulse*, $\delta(t - kT)$, scaled by the area $f(kT)t_w$. Thus, we can use Eq. (2.35) for approximating $f(t)$ by a series of impulses (as shown in Figure 2.16) with $\tau = kT$ and $\Delta\tau = t_w$, and write the following expression for the sampled signal, $f_{tw}{}^*(t)$:

$$f^*_{tw}(t) = \sum_{k=0}^{\infty} f(kT)t_w\delta(t - kT) = t_w \sum_{k=0}^{\infty} f(kT)\delta(t - kT) \tag{8.1}$$

Equation (8.1) denotes the fact that the sampled signal, $f_{tw}{}^*(t)$, is obtained by sampling $f(t)$ at the sampling rate, $1/T$, with pulses of duration t_w. The *ideal sampler* is regarded as the sampler which produces a series of impulses, $f^*(t)$, weighted by the input value, $f(kT)$ as follows:

$$f^*(t) = \sum_{k=0}^{\infty} f(kT)\delta(t - kT) \tag{8.2}$$

Clearly, the *ideal sampler* output, $f^*(t)$, does not depend upon t_w, which is regarded as a *characteristic* of the *real sampler* described by Eq. (8.1). The ideal sampler is thus a real sampler with $t_w = 1$ second.

Since the sampling process gives a non-zero value of $f^*(t)$ *only* for the duration for which the switch remains closed, there are repeated *gaps* in $f^*(t)$ of approximately T seconds, in which $f^*(t)$ is zero. The *holding* process is an *interpolation* of the sampled input, $f^*(t)$, in each time interval, T, so that the gaps are filled. The simplest holding process is the *zero-order hold* (z.o.h.), which holds the input constant over each time interval, T (i.e. applies a zero-order interpolation to $f^*(t)$). As a result,

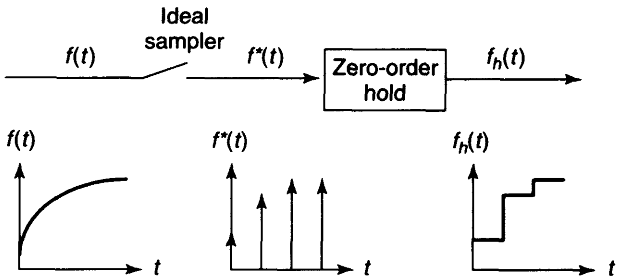

Figure 8.2 Schematic diagram and input, sampled, and output signals of an analog-to-digital (A/D) converter with an ideal sampler and a zero-order hold

the held input, $f_h(t)$, has a staircase time plot. A block diagram of the *ideal sampling* and *holding* processes with a zero-order hold, and their input and output signals, are shown in Figure 8.2.

Since the input to the z.o.h. is a series of impulses, $f^*(t)$, while the output, $f_h(t)$, is a series of steps with amplitude $f(kT)$, it follows that the *impulse response*, $h(t)$, of the z.o.h. must be a *step* that starts at $t = 0$ and ends at $t = T$, and is represented as follows:

$$h(t) = u_s(t) - u_s(t - T) \tag{8.3}$$

or, taking the Laplace transform of Eq. (8.2), the transfer function of the z.o.h. is given by

$$G(s) = (1 - e^{-Ts})/s \tag{8.4}$$

Note that we have used a special property of the Laplace transform in Eq. (8.3) called the *time-shift property*, which is denoted by $\mathcal{L}[y(t - T)] = e^{-Ts}\mathcal{L}[y(t)]$. Similarly, taking the Laplace transform of Eq. (8.2), we can write the Laplace transform of the ideally sampled signal, $F^*(s)$, as follows:

$$F^*(s) = \sum_{k=0}^{\infty} f(kT)e^{-kTs} \tag{8.5}$$

If we define a variable z such that $z = e^{Ts}$, we can write Eq. (8.5) as follows:

$$F(z) = \sum_{k=0}^{\infty} f(kT)z^{-k} \tag{8.6}$$

In Eq. (8.6), $F(z)$ is called the *z-transform* of $f(kT)$, and is denoted by $z\{f(kT)\}$. The z-transform is more useful in studying digital systems than the Laplace transform, because the former incorporates the sampling interval, T, which is a characteristic of digital systems. The expressions for digital transfer functions in terms of the z-transform are easier to manipulate, as they are free from the time-shift factor, e^{-Ts}, of the Laplace transform. In a manner similar to the Laplace transform, we can derive z-transforms of some frequently encountered functions.

Example 8.1

Let us derive the z-transform of $f(kT) = u_s(kT)$, the unit step function. Using the definition of the z-transform, Eq. (8.6), we can write

$$F(z) = \sum_{k=0}^{\infty} u_s(kT)z^{-k} = \sum_{k=0}^{\infty} z^{-k} = 1 + z^{-1} + z^{-2} + z^{-3} + \cdots$$
$$= 1/(1 - z^{-1}) \tag{8.7}$$

or

$$F(z) = z/(z-1) \tag{8.8}$$

Thus, the z-transform of the unit step function, $u_s(kT)$, is $z/(z-1)$. (Note that we have used the *binomial series expansion* in Eq. (8.7), given by $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots$).

Example 8.2

Let us derive the z-transform of the function $f(kT) = \mathrm{e}^{-akT}$, where $t \geq 0$. Using the definition of the z-transform, Eq. (8.6), we can write

$$F(z) = \sum_{k=0}^{\infty} \mathrm{e}^{-akT} z^{-k} = \sum_{k=0}^{\infty} (z\mathrm{e}^{aT})^{-k} = 1 + (z\mathrm{e}^{aT})^{-1} + (z\mathrm{e}^{aT})^{-2} + \cdots$$
$$= 1/[1 - (z\mathrm{e}^{aT})^{-1}] = z/(z - z\mathrm{e}^{-aT}) \tag{8.9}$$

Thus, $z\{\mathrm{e}^{-akT}\} = z/(z - z\mathrm{e}^{-aT})$.

The z-transforms of other commonly used functions can be similarly obtained by manipulating series expressions involving z, and are listed in Table 8.1.

Some important properties of the z-transform are listed below, and may be verified by using the definition of the z-transform (Eq. (8.6)):

(a) Linearity:

$$z\{af(kT)\} = az\{f(kT)\} \tag{8.10}$$

$$z\{f_1(kT) + f_2(kT)\} = z\{f_1(kT)\} + z\{f_2(kT)\} \tag{8.11}$$

(b) Scaling in the z-plane:

$$z\{\mathrm{e}^{-akT} f(kT)\} = F(\mathrm{e}^{aT} z) \tag{8.12}$$

(c) Translation in time:

$$z\{f(kT + T)\} = zF(z) - zf(0^-) \tag{8.13}$$

where $f(0^-)$ is the *initial value* of $f(kT)$ for $k = 0$. Note that if $f(kT)$ has a *jump* at $k = 0$ (such as $f(kT) = u_s(kT)$), then $f(0^-)$ is *understood* to be the value of $f(kT)$

Table 8.1 Some commonly encountered z-transforms

Discrete Time Function, $f(kT)$	z-transform, $F(z)$	Laplace Transform of Equivalent Analog Function, $F(s)$
$u_s(kT)$	$z/(z-1)$	$1/s$
kT	$Tz/(z-1)^2$	$1/s^2$
e^{-akT}	$z/(z-e^{-aT})$	$1/(s+a)$
$1-e^{-akT}$	$z(1-e^{-aT})/[(z-1)(z-e^{-aT})]$	$a/[s(s+a)]$
kTe^{-akT}	$Tze^{-aT}/(z-e^{-aT})^2$	$1/(s+a)^2$
$kT-(1-e^{-akT})/a$	$Tz/(z-1)^2 - z(1-e^{-aT})/[a(z-1)(z-e^{-aT})]$	$a/[s^2(s+a)]$
$\sin(akT)$	$z\sin(aT)/[z^2-2z\cos(aT)+1]$	$a/(s^2+a^2)$
$\cos(akT)$	$z[z-\cos(aT)]/[z^2-2z\cos(aT)+1]$	$s/(s^2+a^2)$
$e^{-akT}\sin(bkT)$	$ze^{-aT}\sin(bT)/[z^2-2ze^{-aT}\cos(bT)-e^{-2aT}]$	$b/[(s+a)^2+b^2]$
$e^{-akT}\cos(bkT)$	$[z^2-ze^{-aT}\cos(bT)]/[z^2-2ze^{-aT}\cos(bT)-e^{-2aT}]$	$(s+a)/[(s+a)^2+b^2]$
$(kT)^n$	$\lim_{a\to 0}(-1)^n d^n/da^n[z/(z-e^{-aT})]$	$n!/s^{n+1}$

before the jump. Thus, for $f(kT)=u_s(kT)$, $f(0^-)=0$. A *negative* translation in time is given by $z\{f(kT-T)\}=z^{-1}F(z)+zf(0^-)$.

(d) Differentiation with z:

$$z\{kTf(kT)\} = -TzdF(z)/dz = -TzF^{(1)}(z) \tag{8.14}$$

(e) Initial value theorem:

$$f(0^+) = \lim_{z\to\infty} F(z) \tag{8.15}$$

Equation (8.15) holds if and only if the said limit exists. Note that if $f(kT)$ has a *jump* at $k=0$ (such as $f(kT)=u_s(kT)$), then $f(0^+)$ is *understood* to be the value of $f(kT)$ *after* the jump. Thus, for $f(kT)=u_s(kT)$, $f(0^+)=1$.

(f) Final value theorem:

$$f(\infty) = \lim_{z\to 1}(1-z^{-1})F(z) \tag{8.16}$$

Equation (8.16) holds if and only if the said limit exists. Using Table 8.1, and the properties of the z-transform, we can evaluate z-transforms of rather complicated functions.

Example 8.3

Let us find the z-transform of $f(kT)=10e^{-akT}\sin(bkT-2T)$. From Table 8.1, we know that

$$z\{\sin(bkT)\} = z\sin(bT)/[z^2-2z\cos(bT)+1] \tag{8.17}$$

Then, using the linearity property of the z-transform given by Eq. (8.10), we can write

$$z\{10\sin(bkT)\} = 10z\sin(bT)/[z^2-2z\cos(bT)+1] \tag{8.18}$$