

Furthermore, the scaling in the z -plane given by Eq. (8.12) yields

$$z\{10e^{-akT} \sin(bkT)\} = 10ze^{aT} \sin(bT)/[z^2e^{2aT} - 2ze^{aT} \cos(bT) + 1] \quad (8.19)$$

Finally, the translation in time given by Eq. (8.13) yields (noting that $10e^0 \sin(0) = 0$)

$$z\{10e^{-akT} \sin(bkT - 2T)\} = 10z^{-1}e^{aT} \sin(bT)/[z^2e^{2aT} - 2ze^{aT} \cos(bT) + 1] \quad (8.20)$$

8.3 Pulse Transfer Functions of Single-Input, Single-Output Systems

In a manner similar to the transfer function in Laplace domain for a single-input, single-output analog system, we can define a *pulse transfer function* for a single-input, single-output digital system in the z -domain as the z -transform of the *output* signal *divided* by the z -transform of the *input* signal. Finding the pulse transfer functions of digital systems in the z -domain is a useful application of the z -transform. However, before finding the pulse transfer function of a digital system, we must distinguish between an *inherently digital system*, and an analog system *rendered digital* due to the process of *data sampling*, as shown in Figure 8.3. It is clear from Figure 8.3 that while an inherently digital system would always produce a *discrete time output*, an analog system would produce a *continuous time output*, even though the input is a digitally sampled signal (recall the continuous time outputs to impulse inputs of analog systems calculated in Chapters 2 and 4).

Finding an *inherently digital* system's pulse transfer function is straightforward, because both input and output are discrete time signals with the same interval, T . On the other hand, taking the z -transform of the transfer function, $G(s)$, of a *sampled-data analog* system – in which the input is *discrete* with a sampling interval, T , while the output is a *continuous* time signal – is quite problematic. However, if we assume that we are *only* interested in finding the output of a sampled-data analog system at the *same*

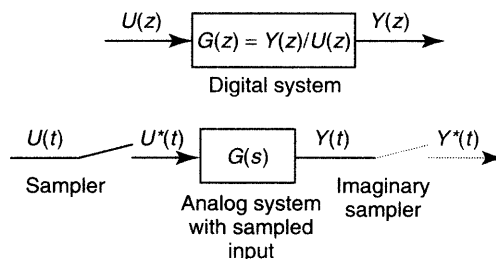


Figure 8.3 Block diagrams of an inherently digital system and a sampled-data analog system rendered digital by an *imaginary sampler* at the output

discrete time points at which the input is sampled, the difficulty in finding the transfer function in the z -domain can be removed. By making such an assumption, we would be disregarding the continuous nature of the output of a sampled-data analog system, and treat the system as if its output is sampled by an *imaginary sampler*. By having such an imaginary sampler at the output, the sampled-data analog system effectively becomes an inherently digital system. The ideally sampled input of an analog system is given by Eq. (8.2) as follows:

$$u^*(t) = \sum_{n=0}^{\infty} u(nT)\delta(t - nT) \quad (8.21)$$

The continuous time output, $y(t)$, is given by the *discretized convolution integral* of Eq. (2.119) as follows:

$$y(t) = \sum_{n=0}^{\infty} u(nT)g(t - nT) \quad (8.22)$$

where $g(t - nT)$ is the *impulse response* of the analog system to a unit impulse applied at time $t = nT$. The digitized output, $y^*(t)$, resulting from sampling of $y(t)$ by an imaginary sampler shown in Figure 8.3, with the same sampling interval, T , as that of the input, $u^*(t)$, can be written as follows:

$$y^*(t) = y(kT) = \sum_{n=0}^{\infty} u(nT)g(kT - nT) \quad (8.23)$$

Then, the z -transform of the (imaginary) sampled output, $y^*(t) = y(kT)$, can be written as

$$Y(z) = z\{y(kT)\} = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} u(nT)g(kT - nT)z^{-k} \quad (8.24)$$

or, expressing the double summation as a product of two summations,

$$Y(z) = \sum_{k=0}^{\infty} g(kT - nT)z^{-k+n} \sum_{n=0}^{\infty} u(nT)z^{-n} \quad (8.25)$$

Since $g(kT - nT) = 0$ for $k < n$, we can begin the first summation in Eq. (8.25) at $k = n$ instead of $k = 0$, and defining $m = k - n$, we can write

$$Y(z) = \sum_{m=0}^{\infty} g(mT)z^{-m} \sum_{n=0}^{\infty} u(nT)z^{-n} \quad (8.26)$$

or, recognizing the second summation of Eq. (8.26) as the z -transform of the input, $U(z)$,

$$Y(z) = \left[\sum_{m=0}^{\infty} g(mT)z^{-m} \right] U(z) \quad (8.27)$$

Comparing Eq. (8.27) with the definition of the pulse transfer function, $G(z) = Y(z)/U(z)$, we can write the pulse transfer function of a sampled-data analog system as follows:

$$G(z) = \sum_{m=0}^{\infty} g(mT)z^{-m} \quad (8.28)$$

Equation (8.28) indicates that the pulse transfer function of a sampled-data analog system is the z -transform of the *digitized* impulse response, $g(mT)$. This is an important result, which shows that although the output, $y(t)$, of a sampled-data analog system is continuous in time, by having an imaginary sampler at the output (Figure 8.3), the analog system essentially becomes a digital system with the pulse transfer function, $G(z)$. Since the impulse response, $g(t)$, of the analog system is the inverse Laplace transform of its transfer function, $G(s)$, we can directly obtain $G(z)$ from $G(s)$ by using Table 8.1, and denote $G(z)$ by the convenient notation, $G(z) = z\{G(s)\}$.

Example 8.4

Let us find the pulse transfer function of the z.o.h. The transfer function of z.o.h is given by Eq. (8.4) as $G(s) = (1 - e^{-Ts})/s$. We can express $G(s)$ as $G(s) = 1/s - e^{-Ts}/s$, and write

$$G(z) = z\{1/s\} - z\{e^{-Ts}/s\} \quad (8.29)$$

From Table 8.1, we note that

$$z\{1/s\} = z/(z - 1) \quad (8.30)$$

Furthermore, note that e^{-Ts}/s is the Laplace transform of $u_s(t - T)$. Therefore, using the time-shift property of the z -transform given by Eq. (8.13), we can write

$$z\{e^{-Ts}/s\} = z\{u_s(kT - T)\} = z^{-1} \cdot z\{u_s(kT)\} = 1/(z - 1) \quad (8.31)$$

Substituting Eqs. (8.30) and (8.31) into Eq. (8.29), we can write the pulse-transfer function of the z.o.h as follows:

$$G(z) = z/(z - 1) - 1/(z - 1) = (z - 1)/(z - 1) = 1 \quad (8.32)$$

Thus, the pulse transfer function of the z.o.h is unity.

Example 8.5

Let us find the pulse transfer function of a sampled-data analog system obtained by placing the z.o.h in *series* with an analog system with transfer function $G_1(s) = s/(s^2 + 1)$. The overall transfer function of the resulting sampled-data analog system is written as

$$G(s) = G_1(s)(1 - e^{-Ts})/s = (1 - e^{-Ts})/(s^2 + 1) \quad (8.33)$$

For finding the pulse transfer function, $G(z)$, we first express $G(s)$ as follows:

$$G(s) = 1/(s^2 + 1) - e^{-Ts}/(s^2 + 1) \quad (8.34)$$

Note that $1/(s^2 + 1)$ is the Laplace transform of $\sin(kT)$. Furthermore, $e^{-Ts}/(s^2 + 1)$ is the Laplace transform of $\sin(kT - T)$. Hence, using the time-shift property of the z -transform and Eq. (8.34), we can write

$$G(z) = z\{\sin(kT)\} - z^{-1} \cdot z\{\sin(kT)\} = (1 - z^{-1}) \cdot z\{\sin(kT)\} \quad (8.35)$$

or, using Table 8.1,

$$\begin{aligned} G(z) &= (1 - z^{-1})z \sin(T)/[z^2 - 2z \cos(T) + 1] \\ &= (z - 1) \sin(T)/[z^2 - 2z \cos(T) + 1] \end{aligned} \quad (8.36)$$

Examples 8.4 and 8.5 indicate that we can obtain the pulse transfer function of a *general* sampled-data analog system consisting of the z.o.h in series with a system with *proper* transfer function, $G_1(s)$, by expressing the overall transfer function as $G(s) = (1 - e^{-Ts})[G_1(s)/s]$. Then, using the time-shift property of the z -transform, and noting that $s_1(t) = \mathcal{L}^{-1}[G_1(s)/s]$ is the *step response* of the system, $G_1(s)$, with $s(0^-) = 0$, we can write the pulse transfer function, $G(z)$, as follows:

$$G(z) = (1 - z^{-1}) \cdot z\{G_1(s)/s\} = (1 - z^{-1}) \cdot z\{s_1(kT)\} \quad (8.37)$$

In Chapter 2, we learnt how to calculate the step response, $s_1(t)$, of a proper transfer function, $G_1(s)$, using the *partial fraction expansion* of $G_1(s)/s$, either by hand, or by the M-file *stepresp.m*. Note that in the M-file *stepresp.m*, the step response, $s_1(t)$, is essentially computed at *discrete* time points, with time interval, $dt = T$. Hence, *stepresp.m* gives us the *digitized step response*, $s_1(kT)$, which can be thought of as the result of sampling $s_1(t)$ by an imaginary sampler (Figure 8.3) with a specified sampling interval, T . If only we can compute the z -transform of a discrete time signal, $s_1(kT)$, by an appropriate M-file, we will have all the necessary tools to compute the pulse transfer function of a general sampled-data system. For the moment, let us confine ourselves to hand calculation of the pulse transfer function.

Example 8.6

Let us find the pulse transfer function of the sampled-data analog system consisting of an A/D converter with z.o.h in *closed-loop* with an analog plant of transfer function, $G_1(s)$, as shown in Figure 8.4(a), if $G_1(s) = (s + 1)/[(s + 2)(s + 3)]$.

To find the pulse transfer function of the closed-loop sampled-data analog system, $G(z) = Y(z)/Y_d(z)$, we have to express the block diagram of the system in such a way that both $y(t)$ and $y_d(t)$ are digital signals. This is done in Figure 8.4(b) by moving the sampler at the input of z.o.h to outside the feedback loop before

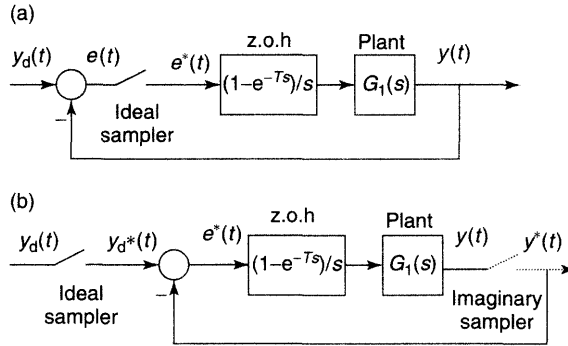


Figure 8.4 Two equivalent block diagrams of closed-loop sampled-data analog system with zero-order hold and analog plant, $G_1(s)$

the summing junction, as well as placing an imaginary sampler in the feedback loop. In doing so, we are not changing the essential characteristic of the closed-loop system. Hence, the block-diagrams shown in Figures 8.4(a) and (b) are *equivalent*. The *open-loop* transfer function of the system is given by

$$G_o(s) = (1 - e^{-Ts})G_1(s)/s \quad (8.38)$$

The pulse transfer function of the closed-loop system is given by $G(z) = Y(z)/Y_d(z)$. From the block-diagram of Figure 8.4(b) it is clear that $e^*(t) = y_d^*(t) - y^*(t)$, or, taking the z -transform, $E(z) = Y_d(z) - Y(z)$. Furthermore, the open-loop pulse transfer function is $z\{G_o(s)\} = G_o(z) = Y(z)/E(z)$. Therefore, the closed-loop pulse transfer function is $G(z) = Y(z)/Y_d(z) = G_o(z)/[1 + G_o(z)]$, where $G_o(z)$ is the pulse transfer function of the open-loop system. We can calculate $G_o(z)$ by finding the z -transform of $G_o(s)$ by Eq. (8.37) as follows:

$$G_o(z) = z\{G_o(s)\} = (1 - z^{-1}) \cdot z\{G_1(s)/s\} \quad (8.39)$$

where

$$\begin{aligned} z\{G_1(s)/s\} &= z\{1/(6s) + 1/[2(s+2)] - 2/[3(s+3)]\} \\ &= z/[6(z-1)] + z/[2(z-e^{-2T})] - 2z/[3(z-e^{-3T})] \end{aligned} \quad (8.40)$$

Substituting Eq. (8.40) into Eq. (8.39) and simplifying, we get

$$\begin{aligned} G_o(z) &= [z(1 + 3e^{-2T} - 4e^{-3T}) + e^{-5T} + 3e^{-3T} \\ &\quad - 4e^{-2T}]/[6(z - e^{-2T})(z - e^{-3T})] \end{aligned} \quad (8.41)$$

Finally, the pulse transfer function of the closed-loop system is obtained as follows:

$$\begin{aligned} G(z) &= G_o(z)/[1 + G_o(z)] \\ &= [z(1 + 3e^{-2T} - 4e^{-3T}) + e^{-5T} + 3e^{-3T} - 4e^{-2T}]/[6(z - e^{-2T})(z - e^{-3T}) \\ &\quad + z(1 + 3e^{-2T} - 4e^{-3T}) + e^{-5T} + 3e^{-3T} - 4e^{-2T}] \end{aligned} \quad (8.42)$$