corresponds to $|G(e^{i\omega T})| = 1$, or $|e^{i\omega T}/(e^{2i\omega T} + 2)| = 1$, which is seen in Figure 8.5 to occur at $\omega T = 1.5708 = \pi/2$. Thus, the value of z at which to 0 dB and the phase is 90° is $z = e^{i\pi/2} = i$, and the corresponding value of $G(z) = G(i) = i$. Note that both Nyquist frequency, $\omega = \pi/T$, and the frequency, $\omega = \pi/(2T)$, at which the gain peaks to 0 dB with phase 90°, will be modified if we change the sampling interval, T.

8.5 Stability of Single-Input, Single-Output Digital Systems

The non-zero sampling interval, T, of digital systems crucially affects their *stability*. In Chapter 2, we saw how the location of *poles* of an analog system (i.e. roots of the denominator polynomial of transfer function, $G(s)$) determined the system's stability. If none of the poles lie in the right-half s-plane, then the system is stable. It would be interesting to find out what is the appropriate location of a *digital* system's *poles* – defined as the roots of the denominator polynomial of the *pulse transfer function*, $G(z)$ – for stability. Let us consider a *mapping* (or *transformation*) of the Laplace domain on to the z-plane. Such a mapping is defined by the transformation, $z = e^{Ts}$. The region of *instability*, namely the right-half s-plane, can be denoted by expressing $s = \alpha + i\omega$, where $\alpha > 0$. The corresponding region in the z-plane can be obtained by $z = e^{(\alpha+i\omega)T} = e^{\alpha T}e^{i\omega T} = e^{\alpha T}[\cos(\omega T) + i\sin(\omega T)]$, where $\alpha > 0$. Note that a *line* with $\alpha =$ constant in the s-plane corresponds to a *circle* $z = e^{\alpha T}[\cos(\omega T) + i\sin(\omega T)]$ in the z-plane of *radius* $e^{\alpha T}$ *centered at* $z = 0$. Thus, a line with $\alpha =$ constant in the right-half s-plane ($\alpha > 0$) corresponds to a circle in the z-plane of *radius larger than unity* ($e^{\alpha T} > 1$). In other words, the right-half s-plane corresponds to the region *outside* a *unit circle* centered at the origin in the z-plane. Therefore, stability of a digital system with pulse transfer function, $G(z)$, is determined by the location of the poles of $G(z)$ with respect to the unit circle in the z-plane. If any pole of $G(z)$ is located *outside* the unit circle, then the system is *unstable*. If all the poles of $G(z)$ lie *inside* the unit circle, then the system is *asymptotically stable*. If some poles lie *on* the unit circle, then the system is *stable*, but *not* asymptotically stable. If poles on the unit circle are *repeated*, then the system is *unstable*.

Example 8.8

Let us analyze the stability of a digital system with pulse transfer function, $G(z) = z/[(z - e^{-T})(z^2 + 1)]$, if the sampling interval is $T = 0.1$ second. The poles of $G(z)$ are given by $z = e^{-T}$, and $z = \pm i$. With $T = 0.1$ s, the poles are $z = e^{-0.1} = 0.9048$, and $z = \pm i$. Since none of the poles of the system lie outside the unit circle centered at $z = 0$, the system is *stable*. However, the poles $z = \pm i$ are *on* the unit circle. Therefore, the system is *not* asymptotically stable. Even if the sampling interval, T, is made very small, the pole at $z = e^{-T}$ will approach – but remain *inside* – the unit circle. Hence, this system is stable for all possible values of the sampling interval, T.

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Example 8.9

Let us find the *range* of sampling interval, T, for which the digital system with pulse transfer function, $G(z) = 100/(z^2 - 5e^{-T} + 4)$, is stable. The poles of $G(z)$ are given by the solution of the characteristic equation $z^2 - 5e^{-T} + 4 = 0$, or $z = \pm(5e^{-T} - 4)^{1/2}$. If $T > 0.51$, then $|z| > 1.0$, indicating instability. Hence, the system is stable for $T \leq 0.51$ second.

Example 8.10

A sampled-data closed-loop system shown in Figure 8.4 has a plant transfer function, $G_1(s) = s/(s^2 + 1)$. Let us find the range of sampling interval, T, for which the closed-loop system is stable.

In Example 8.6, we derived the pulse transfer function of the closed-loop system to be the following:

$$G(z) = [z(1 + 3e^{-2T} - 4e^{-3T}) + e^{-5T} + 3e^{-3T} - 4e^{-2T}]/[6(z - e^{-2T})(z - e^{-3T}) + z(1 + 3e^{-2T} - 4e^{-3T}) + e^{-5T} + 3e^{-3T} - 4e^{-2T}] \tag{8.43}$$

The closed-loop poles are the roots of the following characteristic polynomial:

$$6(z - e^{-2T})(z - e^{-3T}) + z(1 + 3e^{-2T} - 4e^{-3T}) + e^{-5T} + 3e^{-3T} - 4e^{-2T} = 0 \tag{8.44}$$

or

$$6z^2 + z(1 - 3e^{-2T} - 10e^{-3T}) + 7e^{-5T} + 3e^{-3T} - 4e^{-2T} = 0 \tag{8.45}$$

An easy way of solving Eq. (8.45) is through the MATLAB command *roots* as follows:

```
>>T=0.0001;z=roots([6 1-3*exp(-2*T)-10*exp(-3*T) 7*exp(-5*T)+3*exp(-3
  *T)-4*exp(-2*T)]) <enter>

z =
  0.9998
  0.9996
```

Since the poles of $G(z)$ approach the unit circle only in the limit $T \to 0$, it is clear that the system is stable for all non-zero values of the sampling interval, T.

In a manner similar to stability analysis of the closed-loop analog, single-input, single-output system shown in Figure 2.32 using the *Nyquist plot* of the open-loop transfer function, $G_o(s) = G(s)H(s)$, we can obtain the Nyquist plots of *closed-loop digital systems* with an *open-loop* pulse transfer function, $G_o(z)$, using the MATLAB (CST) command *dnyquist* as follows:

```
>>dnyquist(num,den,T) <enter>
```

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where *num* and *den* are the numerator and denominator polynomials of the system's *open-loop* pulse transfer function, $G_o(z)$, in decreasing powers of z, and T is the sampling interval. The result is a *Nyquist plot* (i.e. *mapping* of the *imaginary axis* of the Laplace domain in the $G_o(z)$ plane) on the screen. The user can supply a vector of desired frequency points, w, at which the Nyquist plot is to be obtained, as the fourth input argument of the command *dnyquist*. The command *dnyquist* obtains the Nyquist plot of the digital system in the $G_o(z)$ plane. Hence, the stability analysis is carried out in *exactly the same* manner as for analog systems, using the Nyquist stability theorem of Chapter 2. According to the Nyquist stability theorem for digital systems, the closed-loop system given by the pulse transfer function, $G_o(z)/[1 + G_o(z)]$, is *stable* if the Nyquist plot of $G_o(z)$ encircles the point -1 *exactly* P times in the *anti-clockwise* direction, where P is the number of *unstable* poles of $G_o(z)$.

Example 8.11

Let us use the digital Nyquist plot to analyze the stability of the closed-loop sampled data analog system shown in Figure 8.4 with the plant, $G_1(s) = 1/[(s(s+1)]$. The open-loop transfer function of the system is $G_o(s) = (1 - e^{-Ts})/[s^2(s+1)]$, while the open-loop *pulse transfer function* can be written as follows:

$$G_o(z) = [z(T - 1 + e^{-T}) + 1 - e^{-T} - Te^{-T}]/[(z-1)(z - e^{-T})] \quad (8.46)$$

For a specific value of the sampling interval, such as $T = 0.1$ second, the Nyquist plot is obtained as follows:

```
>>T=0.1;num = [T-1+exp(-T) 1-exp(-T)-T*exp(-T)],den = conv([1
  -1],[1 -exp(-T)]) <enter>

num =
  0.0048   0.0047
den =
  1.0000  -1.9048  0.9048
```

The poles of $G_o(z)$ are obtained as follows:

```
>>roots(den) <enter>

ans =
  1.0000
  0.9048
```

Thus, $G_o(z)$ does not have any poles outside the unit circle, hence $P = 0$. The digital Nyquist plot is obtained in the frequency range 1–10 rad/s as follows:

```
>>w=logspace(0,1);dnyquist(num,den,T,w) <enter>
```

The resulting Nyquist plot is shown in Figure 8.6. Note that the Nyquist plot of $G_o(z)$ *does not* encircle the point -1 in the $G_o(z)$ plane. Hence, the closed-loop system with pulse transfer function, $G_o(z)/[1 + G_o(z)]$, is stable for $T = 0.1$

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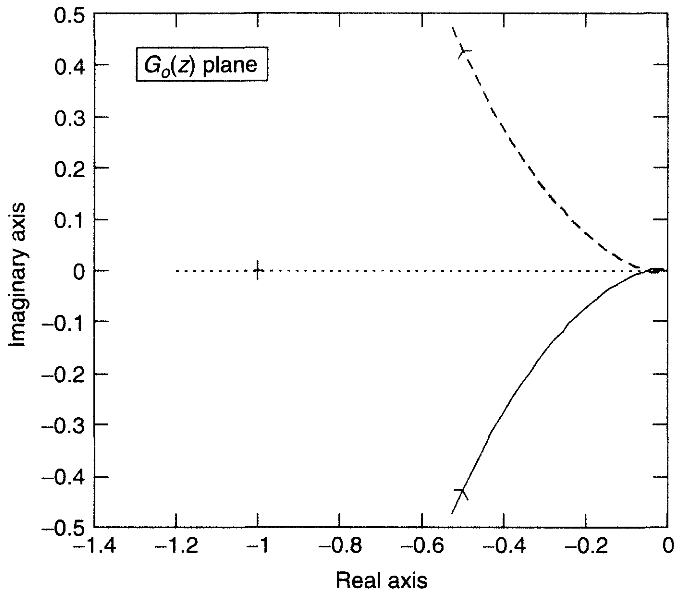

Figure 8.6 Digital Nyquist plot of open-loop pulse transfer function, $G_o(z)$, for the sampled-data closed-loop system of Example 8.10 with sampling interval $T = 0.1$ s

second. We can also analyze the stability of the closed-loop system by checking the closed-loop pole locations as follows:

```
>>sysp=tf(num,den);sysCL=feedback(sysp,1) <enter>

Transfer function:
0.004837z+0.004679
--------------
z^2-1.9z+0.9095


>>[wn,z,p]=damp(sysCL); wn, p <enter>

wn =
 0.9537
 0.9537

p=
 0.9500+0.0838i
 0.9500-0.0838i
```

Note that the returned arguments of the CST command *damp* denote the closed-loop poles, *p*, and the *magnitudes* of the closed-loop poles, *wn*. (*Caution: z* and *wn do not* indicate the *digital* system's damping and natural frequencies, for which you should use the command *ddamp* as shown below.) Since *wn* indicates that both the closed-loop poles are inside the unit circle, the closed-loop system is stable.

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