

Note that we can find the steady state value of response to an *arbitrary* input, $U(z)$, by calculating the digital DC gain of the pulse transfer function $(1 - z^{-1})G(z)U(z)$. Also, an idea about a digital system's response can be obtained from the location of the *dominant* poles with respect to the unit circle in the z -domain. The MATLAB command *ddamp* is quite useful in determining the natural frequencies and damping ratios associated with the poles of a digital system, and is used in a manner similar to the command *damp* for analog systems, with the difference that the sampling interval, T , is specified as the second input argument:

```
>>[mag,wn,z] = ddamp(den,T) <enter>
```

where *den* is the denominator polynomial of the system's pulse transfer function, T is the sampling interval, *mag* is the magnitude of the pole in the z -domain, *wn* is the equivalent natural frequency in the s -plane, and z is the equivalent damping ratio. The command *ddamp* used without the sampling interval as the second input argument gives only the magnitudes of the poles of the system. When used without the output arguments on the left-hand side, the magnitudes, natural frequencies, and damping ratios are printed on the screen. Instead of *den*, you can input the z -plane pole locations as a vector, and get the returned damping ratios and natural frequencies associated with the poles. Let us determine the natural frequencies and damping ratios of the digital system of Example 8.12 when the sampling interval is $T = 0.1$ second as follows:

```
>>ddamp([1 -0.7 0.3],0.1) <enter>
```

Eigenvalue	Magnitude	Equiv. Damping	Equiv. FrEq. (rad/sec)
0.3500+0.4213i	0.5477	0.5657	10.6421
0.3500-0.4213i	0.5477	0.5657	10.6421

A damping ratio of $\zeta = 0.5657$ indicates a well damped response, as seen in Figure 8.7.

For systems of order greater than two, we can identify the dominant poles which are the closest to the unit circle in the z -domain. The performance of such systems is primarily determined by the location of the dominant poles.

8.7 Closed-Loop Compensation Techniques for Single-Input, Single-Output Digital Systems

For improving the performance of digital systems, we can apply closed-loop compensation techniques similar to analog systems discussed in Section 2.12, by placing a compensator with pulse-transfer function, $H(z)$, in closed-loop with a digital plant, $G(z)$, as shown in Figure 8.8. However, we have to be mindful of the dependence of the closed-loop digital system's properties on the sampling interval, T , and the properties of the z -transform, while designing compensators in the z -domain. Let us consider an example of closed-loop digital compensation.

Just as in continuous systems, we can have the ideal *proportional-integral-derivative* (PID) control of digital systems. The digital equivalent of a PID compensator's transfer

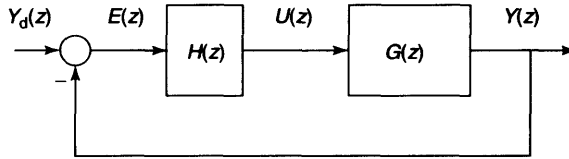


Figure 8.8 A single-input, single-output digital feedback control system with controller pulse transfer function, $H(z)$, and plant pulse transfer function, $G(z)$

function is obtained by combining the *proportional* term with the z -transform of the *derivative* term and the z -transform of the integral term. Note that the time derivative of the error in Figure 8.8, $e^{(1)}(t)$, is approximated by the digital equivalent of a first order *difference* as $e^{(1)}(t) = [e(kT) - e(kT - T)]/T$, whose z -transform is obtained using the time-shift property of z -transform (Eq. (8.13)) as $(1 - z^{-1})E(z)/T$. Thus, the derivative term of PID compensator has the form $K_D(1 - z^{-1})$. The z -transform of the integral term, K_I/s , is $K_I z/(z - 1)$ according to Table 8.1. The PID compensator's pulse transfer is thus given by $H(z) = K_P + K_D(1 - z^{-1}) + K_I z/(z - 1)$, or $H(z) = [(K_P + K_D + K_I)z^2 - (K_P - 2K_D)z + K_D]/(z^2 - z)$.

Example 8.13

Consider a digital system with the following pulse-transfer function and a sampling interval, $T = 0.2$ second:

$$G(z) = 0.0001(z - 1)/(z^2 + 1.97z + 0.99) \quad (8.55)$$

The poles, natural frequencies and damping ratios of the system are obtained as follows:

```
>>ddamp([1 1.97 0.99],0.2) <enter>
```

Eigenvalue	Magnitude	Equiv. Damping	Equiv. FrEq. (rad/sec)
-0.9850+0.1406i	0.9950	0.0017	14.9990
-0.9850-0.1406i	0.9950	0.0017	14.9990

A damping ratio of $\zeta = 0.0017$ indicates a lightly damped response, which is unacceptable. It is desired to increase the closed-loop damping to $\zeta = 0.7$ while keeping the natural frequency unchanged at $\omega_n = 15$ rad/s, through a *proportional-integral* (PI) compensator in the closed-loop configuration of Figure 8.8. With $H(z) = K_P + K_I z/(z - 1)$, the closed-loop pulse-transfer function is given by

$$\begin{aligned} Y(z)/Y_d(z) &= G(z)H(z)/[1 + G(z)H(z)] \\ &= 0.0001[K_P(z - 1) + K_I z]/[z^2 + \{1.97 + 0.0001(K_P + K_I)\}z \\ &\quad + 0.99 - 0.0001K_P] \end{aligned} \quad (8.56)$$

The closed-loop poles are the roots of $z^2 + \{1.97 + 0.0001(K_P + K_I)\}z + 0.99 - 0.0001K_P = 0$, and must be the same as the desired pole locations $z_{1,2} =$

$e^{sT} = \exp(-\zeta \omega_n T) [\cos\{\omega_n T(1 - \zeta^2)^{1/2}\} \pm i \sin\{\omega_n T(1 - \zeta^2)^{1/2}\}]$. Plugging $\zeta = 0.7$ and $\omega_n = 15$ rad/s in the desired characteristic polynomial, $z^2 - 2 \exp(-\zeta \omega_n T) \cos\{\omega_n T(1 - \zeta^2)^{1/2}\}z + \exp(-2\zeta \omega_n T)$, and comparing the coefficients with those of the denominator polynomial Eq. (8.56), we get $K_P = 9750$ and $K_I = -28125$. Let us check whether the closed-loop poles are at desired locations as follows:

```
>>Kp=9750;Ki=-28125;sysGH=tf(0.0001*[Kp+Ki-Kp],[1 1.97 0.99]);
sysCL=feedback(sysGH,1) <enter>
```

Transfer function:

```
-1.838z-0.975
```

```
-----
```

```
z^2+0.1325z+0.015
```

```
>>ddamp([1 0.1325 0.015],0.2) <enter>
```

Eigenvalue	Magnitude	Equiv. Damping	Equiv. Freq.(rad/s)
-6.63e-002+1.03e-001i	1.22e-001	7.00e-001	1.50e+001
-6.63e-002-1.03e-001i	1.22e-001	7.00e-001	1.50e+001

which confirms that the desired closed-loop damping and natural frequency have been achieved.

The steady-state error of the *stable* closed-loop system shown in Figure 8.8 to a *unit step* desired output can be obtained from the final value theorem (Eq. (8.16)) as follows:

$$\begin{aligned} e(\infty) &= \lim_{z \rightarrow 1} (1 - z^{-1})E(z) = \lim_{z \rightarrow 1} (1 - z^{-1})Y_d(z)/[1 + G(z)H(z)] \\ &= \lim_{z \rightarrow 1} 1/[1 + G(z)H(z)] \end{aligned} \quad (8.57)$$

or the steady state error is the digital DC gain of the pulse transfer function $1/[1 + G(z)H(z)]$. For the system of Example 8.13, the steady-state error is computed as follows:

```
>>num=0.0001*[Kp+Ki -Kp];den=[1 1.97 0.99];a=ddcgain(num,den);
e=1/(1+a) <enter>
```

```
e =
3.4510
```

which is very large. The closed-loop steady state error to a step desired output can be reduced to zero by a compensator which has *two* poles at $z = 1$, such as $H(z) = K(z + \alpha)/(z - 1)^2$, since the pole at $z = 1$ of the PI compensator got canceled by the plant zero at the same location. Recall that a pole at $s = 0$ in the s -plane is equivalent to a pole at $z = 1$ in the z -plane; hence, the *type* of a digital closed-loop system is determined by the number of poles at $z = 1$ in the open-loop pulse transfer function, $G(z)H(z)$. For the system of Example 8.13, you can find the compensator, $H(z) = K(z + \alpha)/(z - 1)^2$, as an exercise such that the closed-loop damping and natural frequency are as desired, and the steady-state error to a unit step function is brought to zero.