9

Advanced Topics in Modern Control

9.1 Introduction

The previous chapters saw coverage of topics ranging from classical control to modern digital control. In many modern control applications, certain other techniques are employed which are grouped here as *advanced topics*. A detailed coverage of the advanced topics – such as H_∞ *control*, *input shaping*, and *nonlinear control* – is beyond the scope of the present text. However, you can get a flavor of each topic in this chapter, and for details you are encouraged to look at the many references given at the end of the chapter.

9.2 H_∞ Robust, Optimal Control

In Chapter 6, we derived linear optimal controllers with full-state feedback that minimized a quadratic objective function, and in Chapter 7 we studied Kalman filters as optimal observers in the presence of white noise. The resulting compensator with an optimal regulator (or tracking system) and a Kalman filter was referred to as an LQG (Linear, Quadratic, Gaussian) controller. While LQG controllers exhibit good performance, their robustness to process and measurement noise can only be indirectly ensured by iterative techniques, such as loop-transfer recovery (LTR) covered in Chapter 7. The H_∞ (pronounced *H-infinity*) *optimal control* design technique, however, directly address the problem of robustness by deriving controllers which maintain system response and error signals to within prescribed tolerances, despite the presence of noise in the system. Figure 9.1 shows a plant with transfer matrix, $\mathbf{G}(s)$, input vector, $\mathbf{U}(s)$, and output vector, $\mathbf{Y}(s)$, being controlled by a feedback compensator with transfer matrix, $\mathbf{H}(s)$. The vector $\mathbf{w}(s)$ contains all inputs *external* to the closed-loop system, i.e. process and measurement noise vectors, as well as the desired output vector. The vector $\mathbf{e}(s)$ contains all the *errors* that determine the behavior of the closed-loop system, i.e. the estimation error and the tracking error vectors.

The plant's transfer matrix can be partitioned as follows:

$$\mathbf{G}(s) = \begin{bmatrix} \mathbf{G}_{11}(s) & \mathbf{G}_{12}(s) \\ \mathbf{G}_{21}(s) & \mathbf{G}_{22}(s) \end{bmatrix} \tag{9.1}$$

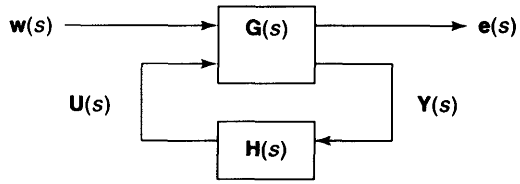

Figure 9.1 Multivariable closed-loop control system with plant, **G**(s), controller, **H**(s), external input vector, **w**(s), error vector, **e**(s), plant input vector, **U**(s), and plant output, **Y**(s)

such that

$$\mathbf{e}(s) = \mathbf{G}_{11}(s)\mathbf{w}(s) + \mathbf{G}_{12}(s)\mathbf{U}(s) \tag{9.2}$$

and

$$\mathbf{Y}(s) = \mathbf{G}_{21}(s)\mathbf{w}(s) + \mathbf{G}_{22}(s)\mathbf{U}(s) \tag{9.3}$$

Substituting the control-law, $\mathbf{U}(s) = \mathbf{H}(s)\mathbf{Y}(s)$, into Eqs. (9.2) and (9.3), we can write the following relationship between the error vector and the external inputs vector:

$$\mathbf{e}(s) = [\mathbf{G}_{11}(s) + \mathbf{G}_{12}(s)\mathbf{H}(s)\{\mathbf{I} - \mathbf{G}_{22}(s)\mathbf{H}(s)\}^{-1}\mathbf{G}_{21}(s)]\mathbf{w}(s) \tag{9.4}$$

The transfer matrix multiplying $\mathbf{w}(s)$ on the right-hand side of Eq. (9.4) can be denoted as $\mathbf{F}(s) = [\mathbf{G}_{11}(s) + \mathbf{G}_{12}(s)\mathbf{H}(s)\{\mathbf{I} - \mathbf{G}_{22}(s)\mathbf{H}(s)\}^{-1}\mathbf{G}_{21}(s)]$ for simplicity of notation, and we can re-write Eq. (9.4) as

$$\mathbf{e}(s) = \mathbf{F}(s)\mathbf{w}(s) \tag{9.5}$$

The H_∞ *optimal control* synthesis procedure consists of finding a *stabilizing* controller, $\mathbf{H}(s)$, such that the H_∞*-norm* of the closed-loop transfer matrix, $\mathbf{F}(s)$, is *minimized.* The H_∞-norm is a *scalar* assigned to a *matrix* (an animal of the same species as the *singular value* discussed in Chapters 2 and 7) and is defined as follows:

$$\|\mathbf{F}(i\omega)\|_\infty = \sup_\omega [\sigma_{\max}(\mathbf{F}(i\omega))] \tag{9.6}$$

where $\sigma_{\max}(\mathbf{F}(i\omega))$ denotes the *largest singular value* of $\mathbf{F}(i\omega)$ (see Chapter 7 for the definition and calculation of singular values), while $\sup_\omega[\cdot]$ is called the *supremum function*, and denotes the *largest value* of the function within the square brackets encountered as the frequency, ω, is varied. Clearly, obtaining the H_∞-norm of a transfer matrix requires calculating the singular values of the transfer matrix with $s = i\omega$ at a range of frequencies, and then obtaining the *maximum* of the *largest singular value* over the given frequency range. Using the MATLAB's *Robust Control Toolbox* command *sigma* for computing the singular values, you can easily write an M-file for calculating the H_∞-norm over a specified frequency range (or *bandwidth*). The LQG problem can be expressed in a similar manner as the *minimization* of another matrix norm, called the H_2*-norm* [1].

To better understand the H_∞ optimal control, let us consider a regulator problem (i.e. a zero desired output) with a *process noise*, $\mathbf{p}(s)$. Then, $\mathbf{w}(s)$ contains only the process noise, and comparing Figures 9.1 and 7.10, we can write $\mathbf{w}(s) = \mathbf{p}(s)$. In Section 7.6, we saw that the *sensitivity* of the *output*, $\mathbf{Y}(s)$, with respect to process noise depends upon the matrix $[\mathbf{I}+\mathbf{G}(s)\mathbf{H}(s)]^{-1}$, which we call the *sensitivity matrix of the output*, $\mathbf{S}(s) = [\mathbf{I}+\mathbf{G}(s)\mathbf{H}(s)]^{-1}$. If a scalar norm of this matrix is minimized, we can be assured of the robustness of the closed-loop system with respect to process noise. Such a scalar norm is the H_∞-norm. Thus, our robust optimal control problem consists of finding a stabilizing controller, $\mathbf{H}(s)$, such that the H_∞-norm of the sensitivity matrix at the output, $\|\mathbf{S}(i\omega)\|_\infty$, is minimized. Note that, in this case, $\mathbf{F}(s) = \mathbf{S}(s)$. However, instead of minimizing $\|\mathbf{S}(i\omega)\|_\infty$ over all frequencies, which will increase sensitivity to high-frequency *measurement noise* and poor stability margins, we should minimize $\|\mathbf{S}(i\omega)\|_\infty$ over only those frequencies where the largest magnitudes of the process noise occur. This is practically achieved by defining a *frequency weighting matrix*, $\mathbf{W}(i\omega)$, such that the *largest singular value* of $\mathbf{W}(i\omega)$ is close to unity in a specified frequency range, $0 \le \omega \le \omega_o$, and rapidly decay to zero for higher frequencies, $\omega > \omega_o$. The frequency, ω_o, is thus regarded as a *cut-off frequency* below which the sensitivity to process noise is to be minimized. The robust, optimal control problem is then solved by finding a stabilizing controller which minimizes $\|\mathbf{W}(i\omega)\mathbf{S}(i\omega)\|_\infty$. The H_∞-optimal control problem for a regulator is thus the *weighted sensitivity minimization* problem. Equating $\mathbf{F}(s) = \mathbf{W}(s)\mathbf{S}(s)$ for the weighted sensitivity minimization, we get the following partition matrices for the *augmented plant*:

$$\mathbf{G}_{11}(s) = \mathbf{W}(s); \quad \mathbf{G}_{12}(s) = -\mathbf{W}(s)\mathbf{G}(s); \quad \mathbf{G}_{21}(s) = \mathbf{I}; \quad \mathbf{G}_{22}(s) = -\mathbf{G} \tag{9.7}$$

which, substituted into Eqs. (9.2) and (9.3), make $\mathbf{e}(s) = \mathbf{W}(s)\mathbf{Y}(s)$.

We learned in Section 7.6 that the requirements of robustness conflict with the requirements of optimal control (i.e. minimal control input magnitudes) and tracking performance (i.e. increased sensitivity to a changing desired output). While robustness to process noise is obtained by minimizing the largest singular value of a frequency weighted sensitivity matrix, $\mathbf{S}(s)$ (as seen above), optimal control requires *minimizing* the *largest* singular value of $\mathbf{H}(s)\mathbf{S}(s)$, and good tracking performance to a changing desired output requires *maximizing* the *smallest* singular value of the *complementary sensitivity matrix*, $\mathbf{T}(s) = \mathbf{I} - \mathbf{S}(s)$. However, high-frequency measurement noise rejection requires *minimizing* the *largest* singular value of $\mathbf{T}(s)$ at high frequencies. Such conflicts can be resolved (as in Section 7.6) by choosing different frequency ranges for maximizing and minimizing the different singular values (or H_∞-norms). The different frequency ranges for the various optimizations can be specified as *three different* frequency weighting matrices, $\mathbf{W}_1(i\omega)$, $\mathbf{W}_2(i\omega)$, and $\mathbf{W}_3(i\omega)$ such that the H_∞-norm of the *mixed-sensitivity matrix*, $\|\mathbf{M}(i\omega)\|_\infty$, is *minimized* where

$$\mathbf{M}(i\omega) = \begin{bmatrix} \mathbf{W}_1(i\omega)\mathbf{S}(i\omega) \\ \mathbf{W}_2(i\omega)\mathbf{H}(i\omega)\mathbf{S}(i\omega) \\ \mathbf{W}_3(i\omega)\mathbf{T}(i\omega) \end{bmatrix} \tag{9.8}$$

Formulating the H_∞-optimal control problem in this fashion ensures the specification of both performance and robustness of the desired closed loop system by the three frequency weighting matrices, such that

$$\sigma_{\max}(\mathbf{S}(i\omega)) \le \sigma_{\max}(\mathbf{W}_1^{-1}(i\omega)) \tag{9.9}$$

$$\sigma_{\max}(\mathbf{H}(i\omega)\mathbf{S}(i\omega)) \le \sigma_{\max}(\mathbf{W}_2^{-1}(i\omega)) \tag{9.10}$$

and

$$\sigma_{\max}(\mathbf{T}(i\omega)) \leq \sigma_{\max}(\mathbf{W}_3^{-1}(i\omega)) \tag{9.11}$$

The mixed-sensitivity H_∞-optimal control problem posed above is difficult to solve for a general system, because the existence of a *stabilizing* solution, $\mathbf{H}(s)$, requires extra conditions [1], apart from those obeyed by the LQG controllers of Chapter 7. However, the advantages of an H_∞ controller lie in its automatic loop shaping as function of the weighting matrices, and hence it is a direct, one-step procedure for addressing both performance and robustness. Here, the weighting matrices are the design parameters to play with. Glover and Doyle [1] provide an efficient algorithm for solving the mixed-sensitivity H_∞-optimal control problem, which results in *two* algebraic Riccati equations. The algorithm imposes the restriction

$$\|\gamma \mathbf{M}(i\omega)\|_\infty \leq 1 \tag{9.12}$$

where γ is a scaling factor, to be determined by the optimization process. The *Robust Control Toolbox* [2] of MATLAB contains the algorithm of Glover and Doyle [1] in an M-file called *hinfopt.m*, which iterates for γ until a stabilizing solution satisfying Eq. (9.12) is obtained.

Example 9.1

Consider the design of an *active flutter-suppression* system for a flexible aircraft wing [3] with the aeroelastic plant's linear, time-invariant model given by the following transfer function:

$$Y(s)/U(s)$$
$$= \frac{[-79.67s^{12} - 2242s^{11} - 4.160s^{10} - 630\,500s^9 - 5.779\times10^6s^8 - 4.007\times10^7s^7 - 1.933\times10^8s^6 - 5.562\times10^8s^5 - 8.377\times10^8s^4 - 6.098\times10^8s^3 - 1.698\times10^8s^2 + 0.0001853s]}{[s^{12} + 39.27s^{11} + 21\,210s^{10} + 4.677\times10^5s^9 + 5.682\times10^6s^8 + 5.233\times10^7s^7 + 3.485\times10^8s^6 + 1.528\times10^9s^5 + 4.289\times10^9s^4 + 7.636\times10^9s^3 + 8.155\times10^9s^2 + 4.683\times10^9s + 1.101x10^9]}$$

The single-input, single-output, 12th order plant has a *trailing-edge control-surface deflection* as the input, $U(s)$, and *normal acceleration* at a sensor location as the output, $Y(s)$. The design objective is to minimize the sensitivity matrix at frequencies below 1 rad/sec, while achieving approximately 20 dB/decade roll-off above 8 rad/sec, for suppression of *flutter*. (*Flutter* is a destructive structural instability of aircraft wings.) In the present case, the weighting matrices are chosen as follows

$$\mathbf{W}_1(s) = (s^2 + 2s + 1)/(s^2 + 60s + 900) \tag{9.13}$$

$$\mathbf{W}_2(s) = (0.01s + 0.1)/(s + 0.1) \tag{9.14}$$

$$\mathbf{W}_3(s) = (0.010\,533s + 3.16)/(0.1s + 1) \tag{9.15}$$

Note that the weighting matrices are scalar functions. The stable solution to the mixed-sensitivity H_∞-optimal control problem is solved by the MATLAB's Robust Control Toolbox [2] function *hinfopt*, which iteratively searches for the optimum value of γ, and is carried out by the following MATLAB statements:

```
>>sysp=ss(sysp); [a,b,c,d]=ssdata(sys); % plant's state-space model
   <enter>

>> w1=[1 2 1;1 60 900]; w2=[0.01 0.1;1 0.1]; w3=[0.010533 3.16;0.1 1];
   % frequency weights <enter>

>>[A,B1,B2,C1,C2,D11,D12,D21,D22]=augtf(a,b,c,d,w1,w2,w3);
   % 2-port augmented system <enter>

>>[gamopt,acp,bcp,ccp,dcp,acl,bcl,ccl,dcl]=hinfopt
   (A,B1,B2,C1,C2,D11,D12,D21,D22); <enter>

<<H-Infinity Optimal Control Synthesis >>
No   Gamma     D11<=1   P-Exist   P>=0   S-Exist   S>=0   lam(PS)<1   C.L.
---------------------------------------------------------------------------
1   1.0000e+000   OK   OK         OK     OK        FAIL   OK          UNST
2   5.0000e-001   OK   OK         OK     OK        FAIL   OK          UNST
3   2.5000e-001   OK   OK         OK     OK         OK    OK          STAB
4   3.7500e-001   OK   OK         OK     OK         OK    OK          STAB
5   4.3750e-001   OK   OK         OK     OK        FAIL   OK          UNST
6   4.0625e-001   OK   OK         OK     OK         OK    OK          STAB
7   4.2188e-001   OK   OK         OK     OK        FAIL   OK          UNST
8   4.1406e-001   OK   OK         OK     OK        FAIL   OK          UNST
9   4.1016e-001   OK   OK         OK     OK        FAIL   OK          UNST

   Iteration no. 6 is your best answer under the tolerance: 0.0100.
```

Hence, the optimum value of $\gamma = 0.40625$ for a *design tolerance* of 0.01, is obtained in nine iterations. A stable closed-loop initial response (Figure 9.2), and a 2.2 percent increase in the flight-velocity at which flutter occurs at standard sea-level are the results of this H_∞ design [3].

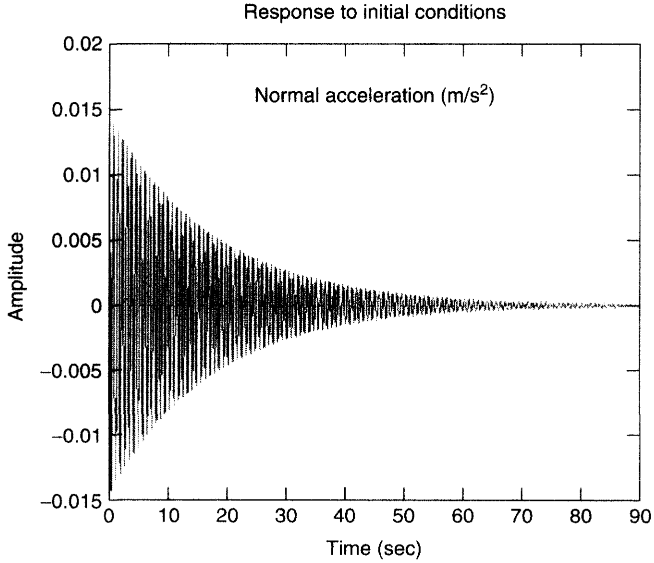

Figure 9.2 Closed-loop initial response of the H_∞-based active flutter suppression system