9.3 Structured Singular Value Synthesis for Robust Control

In Chapter 7, as well as in the previous section, we treated variations in a plant's model – due to factors such as unmodeled dynamics, linearization errors, and parametric uncertainties – as the random *process noise*. However, in certain cases (just as there is a method in madness), it is possible to identify a *structure* in the *uncertainty* in the plant's transfer matrix, $\mathbf{G}(s)$. The plant's *actual transfer matrix*, $\mathbf{G}(s)$, can be then written as

$$\mathbf{G}(s) = \mathbf{G_o}(s) + \Delta_{\mathbf{A}}(s) \tag{9.16}$$

or

$$\mathbf{G}(s) = \mathbf{G_o}(s)[\mathbf{I} + \Delta_{\mathbf{MI}}(s)] \tag{9.17}$$

or

$$\mathbf{G}(s) = [\mathbf{I} + \Delta_{\mathbf{Mo}}(s)]\mathbf{G_o}(s) \tag{9.18}$$

where $\mathbf{G_o}(s)$ is the *nominal plant transfer matrix* (arrived at by certain physical modeling), $\Delta_{\mathbf{A}}(s)$ is the *additive structured uncertainty*, $\Delta_{\mathbf{MI}}(s)$ is the *multiplicative structured uncertainty occurring at the plant's input*, and $\Delta_{\mathbf{Mo}}(s)$ is the *multiplicative structured uncertainty occurring at the plant's output*. Equations (9.16)–(9.18) provide *three* alternative ways in which a plant with structured uncertainty can be expressed. In the LQG/LTR method of Chapter 7 and H_∞-optimal control of Section 9.2, we had assumed that the uncertainty in the plant's model (denoted by the process noise) was random, or *unstructured*. A compensator design made robust with respect to an unstructured uncertainty is unduly conservative, because the uncertainty in a physical plant model is usually structured. It is a task for the designer to identify the structure in the uncertainty ($\Delta_{\mathbf{A}}(s)$, $\Delta_{\mathbf{MI}}(s)$, or $\mathbf{\Delta_{Mo}}(s)$), and make the compensator robust with respect to the structured uncertainty. The most common structured uncertainty model is a *block-diagonal* structure for the additive uncertainty, given by

$$\Delta_{\mathbf{A}}(s) = \mathbf{diag}\{\Delta_{\mathbf{1}}(s), \ldots, \Delta_{\mathbf{m}}(s)\} \tag{9.19}$$

which indicates a matrix with *m blocks*, $\Delta_{\mathbf{1}}(s), \ldots, \Delta_{\mathbf{m}}(s)$, occurring on the *diagonal*. Each block, $\Delta_j(s)$, represents a *different* structure associated with a particular uncertainty, and could be either a scalar or a matrix. By employing a block-diagonal structure, it is assumed that the uncertainty of *each set of elements* of $\mathbf{G_o}(s)$ is independent of the uncertainty for the others.

Robustness of closed-loop systems with respect to the structured uncertainty in the plant is most directly addressed by the *structured singular value*, introduced by Doyle [4]. For a plant with an additive uncertainty matrix, $\mathbf{\Delta_A}(s)$, which is assumed to have a block-diagonal structure given by Eq. (9.19), with each uncertainty block being *stable* and obeying

$$\|\Delta_{\mathbf{j}}(i\omega)\|_\infty \leq \partial \tag{9.20}$$

where ∂ is a selected positive number less than unity.

We can treat the uncertain closed-loop system as the *compensated nominal plant* with transfer matrix, $\mathbf{P_{11}}(s) = \mathbf{G_o}(s)\mathbf{H}(s)[\mathbf{I} + \mathbf{G_o}(s)\mathbf{H}(s)]^{-1}$ (where $\mathbf{H}(s)$ is the compensator's transfer matrix), and an additional *feedback loop* with a *fictitious controller*, $\mathbf{\Delta_A}(s)$. The additional feedback loop consists of the uncertain variables, $\mathbf{U_1}(s)$ and $\mathbf{Y_1}(s)$, that are the input and the output, respectively, of the fictitious controller, $\Delta_\mathbf{A}(s)$. Then we can write the uncertain closed-loop system in the following partitioned form:

$$\begin{bmatrix} \mathbf{Y}(s) \\ \mathbf{U}_1(s) \end{bmatrix} = \begin{bmatrix} \mathbf{P_{11}}(s) & \mathbf{P_{12}}(s) \\ \mathbf{P_{21}}(s) & \mathbf{P_{22}}(s) \end{bmatrix} \begin{bmatrix} \mathbf{Y_d}(s) \\ \mathbf{Y_1}(s) \end{bmatrix} \tag{9.21}$$

The *structured singular value* of the matrix, $\mathbf{P_{22}}(s)$, in the frequency domain is defined as

$$\mu(\mathbf{P_{22}}(i\omega)) = \begin{bmatrix} 0; & \text{if } \det[\mathbf{I} - \mathbf{P_{22}}(i\omega)\Delta_\mathbf{j}(i\omega)] \neq 0 \text{ for any } \Delta_\mathbf{j}(i\omega) \\ 1/\min_j[\sigma_{\max}(\Delta_\mathbf{j}(i\omega)]; & \text{if } \det[\mathbf{I} - \mathbf{P_{22}}(i\omega)\Delta_\mathbf{j}(i\omega)] = 0 \end{bmatrix} \tag{9.22}$$

where $\det[\cdot]$ denotes the determinant. Computation of $\mu(\mathbf{P_{22}}(i\omega))$ is made possible by the following lower and upper bounds [4]:

$$\max_U \rho(\mathbf{UP_{22}}(i\omega)) \leq \mu(\mathbf{P_{22}}(i\omega)) \leq \inf_{\mathrm{D}}[\sigma_{\max}(\mathbf{DP_{22}}(i\omega)\mathbf{D}^{-1})] \tag{9.23}$$

where $\mathbf{U}$ is a *unitary matrix* (i.e. a matrix with the property $\mathbf{UU^H} = \mathbf{I}$) of the *same* block-diagonal structure as $\Delta_\mathbf{A}(s)$, $\mathbf{D}$ is a real, diagonal positive definite matrix, with the following structure:

$$\mathbf{D} = \mathbf{diag}\{d_1\mathbf{I}, \ldots, d_m\mathbf{I}\} \tag{9.24}$$

(where $d_j > 0$), $\inf_D[\cdot]$ is the *infimum function*, and denotes the *minimum value* of the matrix within the square brackets, *with respect to* the matrix $\mathbf{D}$, and $\rho(\mathbf{P})$ denotes the *spectral radius of a square matrix* $\mathbf{P}$, defined as

$$\rho(\mathbf{P}) = \max_j |\lambda_j(\mathbf{P})| \tag{9.25}$$

with $\lambda_j(\mathbf{P})$ denoting an eigenvalue of $\mathbf{P}$. It can be shown using Eq. (9.25) and the definition of the singular values that the spectral radius obeys the following inequality:

$$|\lambda(\mathbf{P})| \leq \rho(\mathbf{P}) \leq \sigma_{\max}(\mathbf{P}) \tag{9.26}$$

Doyle [4] shows that the lower bound in Eq. (9.23) is actually an *equality*, and the upper bound is an equality *if* there are *no more* than three blocks in $\Delta_\mathbf{A}(s)$ with *no repetitions*. Equation (9.26) thus provides *two different* methods of computing $\mu(\mathbf{P_{22}}(i\omega))$, which is, however, a formidable task requiring *nonlinear optimization*.

The objective of the *structural singular value synthesis* (also called *μ-synthesis*) is to find a stabilizing controller, $\mathbf{H}(s)$, and a diagonal scaling matrix, $\mathbf{D}$, such that

$$||\mathbf{DP_{22}}(i\omega)\mathbf{D^{-1}}||_\infty < 1 \tag{9.27}$$

However, it is more practical to assign a mixed-sensitivity frequency weighted control of the type given in the previous section, and replace $\mathbf{P_{22}}(i\omega)$ by the mixed-sensitivity matrix, $\mathbf{M}(i\omega)$, given by Eq. (9.8), such that

$$\|\mathbf{DM}(i\omega)\mathbf{D}^{-1}\|_\infty < 1 \tag{9.28}$$

Since $\|\mathbf{DM}(i\omega)\mathbf{D}^{-1}\|_\infty$ is the upper bound of $\mu(\mathbf{M}(i\omega))$ (Eq. (9.23)), it implies that Eq. (9.28) is sufficient to ensure both stability and performance robustness.

The μ-synthesis problem requires nonlinear optimization, and MATLAB provides the M-file *musyn.m* in the Robust Control Toolbox [2] for solving the combined structured/unstructured uncertainty problem. Also, a dedicated toolbox called the *μ-Analysis and Synthesis Toolbox* [5] is available for use with MATLAB for the computation of structured singular values, as well as for analyzing and designing robust control systems for plants with uncertainty using structured singular values. The iterative procedure used in μ-synthesis is based upon repetitive solution for the stabilizing H_∞ controller, and the associated diagonal scaling matrix, $\mathbf{D}$, which minimizes the structured singular value given by Eq. (9.22).

Example 9.2

Reconsider the active flutter-suppression system of Example 9.1. For μ-synthesis, two alternative choices of the disturbance frequency weight, $\mathbf{W}_1(s)$, are:

$$\mathbf{W_1}(s) = (s^2 + 2s + 1)/(s^2 + 60s + 900) \tag{9.29}$$

and

$$\mathbf{W_1}(s) = (0.01s + 1)/(0.1s + 1) \tag{9.30}$$

while $\mathbf{W_2}(s)$ and $\mathbf{W_3}(s)$ are taken to be zeros. While the two performance specifications are similar at frequencies above the open-loop flutter frequency (8.2 rad/s), their shapes are different at frequencies less than 5 rad/s. Let us begin with the first expression for $\mathbf{W_1}(s)$ and obtain an optimal controller by μ-synthesis, using the Robust Control Toolbox functions *augtf* to construct the two-port augmented state-space plant model, and *musyn* to iterate for μ-synthesis as follows:

```
>>[A,B1,B2,C1,C2,D11,D12,D21,D22]=augtf(a,b,c,d,[1 2 1;1 60 900],[],[]);
   <enter>

>>[acp,bcp,ccp,dcp,mu,logd,ad,bd,cd,dd,gam]=
   musyn(A,B1,B2,C1,C2,D11,D12,D21,D22,logspace(-2,2)); <enter>

       << H-Infinity Optimal Control Synthesis >>
No  Gamma         D11<=1  P-Exist  P>=0  S-Exist  S>=0  lam(PS)<1  C.L.
------------------------------------------------------------------------
1   1.0000e+000   OK      OK       OK    OK       OK    OK         UNST
2   5.0000e-001   OK      FAIL     FAIL  OK       OK    OK         STAB
3   2.5000e-001   OK      OK       OK    OK       OK    OK         UNST
4   1.2500e-001   OK      FAIL     FAIL  OK       OK    OK         UNST
5   6.2500e-002   OK      OK       OK    OK       OK    OK         UNST
6   3.1250e-002   OK      FAIL     OK    OK       OK    OK         UNST
7   1.5625e-002   OK      OK       OK    OK       OK    OK         UNST
```

```
8    7.8125e-003  OK      FAIL    OK      OK     OK     OK          UNST
9    3.9063e-003  OK      OK      OK      OK     OK     OK          UNST
10   1.9531e-003  OK      OK      OK      OK     OK     OK          UNST
11   9.7656e-004  OK      OK      OK      OK     OK     OK          UNST
12   4.8828e-004  OK      FAIL    OK      OK     OK     OK          UNST
13   2.4414e-004  OK      FAIL    FAIL    OK     OK     OK          STAB
14   1.2207e-004  OK      OK      OK      OK     OK     OK          STAB
15   1.8311e-004  OK      OK      OK      OK     OK     OK          UNST
16   1.5259e-004  OK      FAIL    FAIL    OK     OK     OK          UNST
17   1.3733e-004  OK      OK      FAIL    OK     OK     OK          UNST
18   1.2970e-004  OK      OK      OK      OK     OK     OK          UNST
19   1.2589e-004  OK      FAIL    FAIL    OK     OK     OK          STAB
20   1.2398e-004  OK      OK      OK      OK     OK     OK          STAB
21   1.2493e-004  OK      OK      OK      OK     OK     OK          UNST

   Iteration no. 20 is your best answer under the tolerance: 0.0100.

Executing SSV......................Done SSV
```

Hence, the optimum γ value for the first value of $\mathbf{W_1}(s)$ (Eq. (9.29)) is $\gamma = 1.2398 \times 10^{-4}$. Next we design a μ-synthesis controller for the second value of $\mathbf{W_1}(s)$ (Eq. (9.30)) as follows:

```
>>[A,B1,B2,C1,C2,D11,D12,D21,D22]=augtf(a,b,c,d,[0.01 1;0.1 1],[],[]);
   <enter>

>>[acp,bcp,ccp,dcp,mu,logd,ad,bd,cd,dd,gam]=
   musyn(A,B1,B2,C1,C2,D11,D12,D21,D22,logspace(-2,2)); <enter>

<<H-Infinity Optimal Control Synthesis >>
No   Gamma        D11<=1   P-Exist   P>=0   S-Exist   S>=0   lam(PS)<1   C.L.
-------------------------------------------------------------------------------
1    1.0000e+000  OK       OK        OK     OK        OK     OK          UNST
2    5.0000e-001  OK       OK        OK     OK        OK     OK          STAB
3    7.5000e-001  OK       OK        OK     OK        OK     OK          STAB
4    8.7500e-001  OK       OK        FAIL   OK        OK     OK          UNST
5    8.1250e-001  OK       OK        OK     OK        OK     OK          STAB
6    8.4375e-001  OK       FAIL      FAIL   OK        OK     OK          UNST
7    8.2813e-001  OK       OK        FAIL   OK        OK     OK          UNST
8    8.2031e-001  OK       FAIL      FAIL   OK        OK     OK          UNST

   Iteration no. 5 is your best answer under the tolerance: 0.0100.

Executing SSV............................Done SSV
```

Hence, the second design has $\gamma = 0.8125$. Let us compare the structured singular value plots of the two designs in Figure 9.3, by plotting the returned *mu* vectors in dB on a semilog frequency scale. We see that both the designs achieve the same structured singular value at higher frequencies, while at lower frequencies the second design ($\gamma = 0.8125$) produces a slightly smaller value of μ. However, the range of variation of μ is very small in Figure 9.3, which indicates that the two designs converge to the same result.