Input shaping is not limited to linear systems. The time-optimal control principles can be extended to a nonlinear plant for arriving at the pre-shaped inputs [13,14]. Banerjee and Singhose [14] presented an application of time-optimal input shaping to reduce nonlinear vibration in a *two-link flexible robotic manipulator*, while using either a linear or a nonlinear feedback control for tracking. The input shaping was based on two flexible modes. Combining a feedback controller with a feedforward input shaping is a practical method of compensating for the loss of information about nonlinear system dynamics when only a few flexible modes are included in input shaping. However, the implementation of such compensators is greatly dependent on the robustness of the feedback controller. Also, while optimizing for the hybrid feedback/input-shaping controllers, time optimality cannot be guaranteed.

In the several approaches presented in the literature, the adequacy of input shaping for time-optimal control with one or two flexible modes has been demonstrated. Expectedly, by including a larger number of flexible modes, N, a smoother input profile and a reduced transient vibration can be obtained. However, if input profiles must be stored, memory could become a factor when the number of flexible modes is large. The size of the impulse sequence of Eq. (9.39) is 3^N, where N is the number of flexible modes retained in input shaping. In Example 9.3, an input sequence of nine impulse sequence was obtained using two flexible modes.

9.5 Output-Rate Weighted Linear Optimal Control

The optimal linear quadratic regulator (LQR) problem (Chapter 6) is the at the heart of many modern optimal, robust control design methods, such as the linear quadratic Gaussian procedure with loop-transfer recovery (LQG/LTR) (see Chapter 7), and H_∞ control (Section 9.2). The output-weighted LQR problem (referred to as LQRY) (see Section 6.4) minimizes an objective function containing the quadratic form of the measured output as an integrand. However, in several applications it may be more desirable to minimize the *time-rate of change* of the output, rather than the measured output itself. Examples of such active control applications are *vibration reduction* of flexible structures, *flutter-suppression* (Examples 9.1, 9.2), *gust and maneuver load alleviation* of aircraft and *ride-quality augmentation* of any vehicle. In these cases, the measured output is usually the *normal acceleration*, while it is necessary from considerations of passenger/crew comfort (as well as issues such as weapons aiming and delivery) to have an optimal controller that minimizes the *roughness* of the motion, which can be defined as the time-rate of change of normal acceleration. This mechanical analogy can be extended to other physical systems, where sensor limitations prevent the measurement of time-rate of change of an available signal, and even to economic models. Also, it is well known that certain *jerky, nonlinear* motions, if uncorrected, can lead to chaos [15]. Optimal control of such motions may require *output-rate weighted* (ORW) objective functions.

Consider a linear, time-invariant system described by the following state and output equations:

$$\mathbf{x}^{(1)}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{9.42}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \tag{9.43}$$

The regulator design problem is finding an optimal feedback gain matrix, $\mathbf{K}$, which obeys the control-law:

$$\mathbf{u}(t) = -\mathbf{K}\mathbf{x}(t) \tag{9.44}$$

such that the following infinite-time, *output-rate weighted* objective function is minimized:

$$J = \int_0^\infty [\mathbf{y}^{(1)}(t)]^T \mathbf{Q}\mathbf{y}^{(1)}(t) + \mathbf{u}^T(t)\mathbf{R}\mathbf{u}(t)]\,dt \tag{9.45}$$

Equation (9.45) can be re-written as

$$J = \int_0^\infty [\mathbf{x}^{\mathrm{T}}(t)\mathbf{Q}_{\mathrm{c}}\mathbf{x}(t) + \mathbf{x}^{\mathrm{T}}(t)\mathbf{S}^{\mathrm{T}}\mathbf{U}(t) + \mathbf{U}^T(t)\mathbf{S}\mathbf{x}(t) + \mathbf{U}^{\mathrm{T}}(t)\mathbf{R}_{\mathrm{c}}\mathbf{U}(t)]\,dt \tag{9.46}$$

where

$$\mathbf{Q}_{\mathrm{c}} = \mathbf{A}^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}\mathbf{Q}\mathbf{C}\mathbf{A}; \quad \mathbf{S} = \mathbf{D}_{\mathrm{c}}^{\mathrm{T}}\mathbf{Q}\mathbf{C}\mathbf{A}; \quad \mathbf{R}_{\mathrm{c}} = \mathbf{R} + \mathbf{D}_{\mathrm{c}}^{\mathrm{T}}\mathbf{Q}\mathbf{D}_{\mathrm{c}}; \quad \mathbf{D}_{\mathrm{c}} = [\mathbf{CB}; \mathbf{D}];$$
$$\mathbf{U}(t) = [\mathbf{u}(t); \quad \mathbf{u}^{(1)}(t)]^T \tag{9.47}$$

It is to be noted that the output weighted LQRY control (Section 6.4) is a sub-case of the general *output-rate weighted* (ORW) control described by Eqs. (9.42)–(9.47). For *strictly proper* plants ($\mathbf{D} = \mathbf{0}$), the performance integral in Eq. (9.46) reduces to a form similar to of the general LQRY problem. Using steps similar to those of Section 6.4, the optimal regulator gain matrix, $\mathbf{K}$, which minimizes the objective function of Eq. (9.46) subject to Eq. (9.40), is given by

$$\mathbf{K} = \mathbf{R}^{-1}(\mathbf{B}^{\mathrm{T}}\mathbf{M} + \mathbf{S}) \tag{9.48}$$

where $\mathbf{M}$ is the solution of the following *algebraic Riccati equation*:

$$\mathbf{M}\mathbf{A}_{\mathrm{c}} + \mathbf{A}_{\mathrm{c}}{}^{\mathrm{T}}\mathbf{M} - \mathbf{M}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{M} + \mathbf{Q}_{\mathrm{c}} - \mathbf{S}^{\mathrm{T}}\mathbf{R}\mathbf{S} = \mathbf{0} \tag{9.49}$$

in which

$$\mathbf{A}_{\mathrm{c}} = \mathbf{A} - \mathbf{B}\mathbf{R}^{-1}\mathbf{S} \tag{9.50}$$

is the regulator state-dynamics matrix. After a regulator is designed by the above procedure, an observer (or Kalman filter) can be designed with the usual procedure of Chapters 5 or 7, and combined to form a compensator using the separation principle.

Example 9.5

The longitudinal dynamics of the flexible bomber airplane (Example 4.7) is selected as an example to demonstrate output-rate weighted (ORW) optimal control. For the bomber airplane, it is necessary that all transient motion defined by the normal acceleration, $y_1(t)$, and the pitch-rate, $y_2(t)$, should quickly decay to zero when a vertical gust is encountered, since the stability of the bombing platform is crucial for weapons aiming and delivery. The ORW controller minimizes the *time-rate of change of normal acceleration*, $y_1^{(1)}(t)$, and the *pitch-acceleration*, $y_2^{(1)}(t)$. For this strictly proper plant, the best case ORW regulator is obtained by taking $\mathbf{Q} = 10^{-8}\mathbf{I}$ and $\mathbf{R} = \mathbf{I}$, and the optimal ORW regulator gain matrix, $\mathbf{K}$, the solution to the algebraic Riccati equation, $\mathbf{M}$, and the closed-loop eigenvalues, $\mathbf{E}$, are calculated using the MATLAB command *lqry* as follows: