```
>>Dc=C*B; Cc=C*A; [K,M,E]=lqry(A,B,Cc,Dc, 1e-8*eye(2),eye(2))
  <enter>

K=
   0.0297   0.0070   0.0010   0.0001   -0.7041   -0.4182
   0.0557  -0.0074   0.0015  -0.0004   -0.2539   -0.1670

M=
   0.1531   0.0023   0.0044  -0.0002   0.1084    0.0480
   0.0023   0.0036   0.0001   0.0001   -0.0137   -0.0061
   0.0044   0.0001   0.0001   0.0000   0.0029    0.0013
  -0.0002   0.0001   0.0000   0.0000   -0.0006   -0.0003
   0.1084  -0.0137   0.0029  -0.0006   0.1945    0.0871
   0.0480  -0.0061   0.0013  -0.0003   0.0871    0.0390

E=
  -94.2714
  -9.6068
  -1.3166+5.0317i
  -1.3166-5.0317i
  -0.4260+1.8738i
  -0.4260-1.8738i
```

You can compare the closed-loop initial response of the ORW regulator designed above, with that of the traditional LQRY regulator designed with the same weighting matrices, **Q** and **R**. Generally, when applied to flexible structures, the ORW regulator produces a much *smoother response* – with smaller overshoots – which decays faster than that of the corresponding LQRY regulator [16, 17]. The application of ORW optimal control is not limited to flexible structures, and can be extended to any plant where smoothening of the transient response is critical. However, in some applications the sensitivity to noise may increase with ORW based compensators, and must be carefully studied before implementing such controllers.

9.6 Nonlinear Optimal Control

Up to this point, we have confined our attention to the design of linear control systems – such as the classical approach of Chapter 2, the pole-placement state-space design of Chapter 5, and the optimal, linear, state-space design of Chapter 6. In Chapter 2, we had seen how nonlinear systems can be linearized by assuming *small amplitude* motion about an *equilibrium point*. Hence, linear control design techniques can be used for controlling nonlinear plants *linearized* about an equilibrium point. However, when we are interested in *large amplitude* motion of a nonlinear plant – either about an equilibrium point, or *between* two equilibrium points (such as the motion of a pendulum going from the equilibrium point at $\theta = 0^\circ$ to that at $\theta = 180^\circ$) – the linearization of the plant is invalid, and one has to grapple with the nonlinear model of the plant. The control design strategy for a nonlinear plant can be based upon either *linear* or *nonlinear* feedback control laws.

Since we cannot talk about the *poles* of a nonlinear plant, there can be no *nonlinear* pole-placement design approach analogous to the methods of Chapter 5. A possible nonlinear control strategy could be to come-up with a nonlinear feedback control law by *trial and error*, that meets the closed-loop design requirements verified by carrying out a nonlinear simulation (using the techniques of Section 4.6). Surely, such an approach cannot be called a *design strategy*, due to its *ad hoc* nature. An alternative design approach is to *transform* the nonlinear plant into a linear system using an appropriate *feedback* control law, and then treat the linearized system by the *linear control design* strategies covered up to this point. Such a procedure is referred to as *feedback linearization*. Feedback linearization requires a very complicated (generally nonlinear) feedback control law, which is quite sensitive to parametric uncertainties [18]. For a time-varying nonlinear plant, the use of feedback linearization requires time-dependent scheduling of the feedback linearization control law – called *adaptive feedback linearization* [19]. It is clear that the success of feedback linearization is limited to those nonlinear plants which are feedback linearizable. Fortunately, there is another nonlinear control design strategy, called *nonlinear optimal control*, which can be applied generally to control a nonlinear plant.

Nonlinear optimal control is carried out in a manner similar to the linear optimal control of Chapter 6 by minimizing an objective function formulated in terms of the energy of motion and the control input energy. However, nonlinear optimal control – owing to the nonlinear nature of the governing differential equations and control laws – is mathematically more complex than the linear optimal control of Chapter 6. It is beyond the scope of this book to give details of the nonlinear optimal control theory, and you are referred to Kirk [20] and Bellman [21] for the general derivation of the nonlinear optimal control problem.

Let us consider a nonlinear plant with the following state-equation:

$$\mathbf{x}^{(1)}(t) = \mathbf{f}\{\mathbf{x}(t), \mathbf{u}(t), t\} \tag{9.51}$$

where $\mathbf{x}(t)$ is the state-vector, $\mathbf{u}(t)$ is the input vector, and $\mathbf{f}\{\mathbf{x}(t), \mathbf{u}(t), t\}$ denotes a nonlinear vector function involving the state variables, the inputs, and time, t. The solution of Eq. (9.51) with a known input vector, $\mathbf{u}(t)$, was discussed in Section 4.6, with some special conditions to be satisfied by the nonlinear function, $\mathbf{f}\{\mathbf{x}(t), \mathbf{u}(t), t\}$ for the existence of the solution, $\mathbf{x}(t)$, such as the *continuity in time* and the *Lipschitz condition* given by Eq. (4.85). Suppose such conditions are satisfied, and we can solve Eq. (9.51) for $\mathbf{x}(t)$ if we are specified $\mathbf{u}(t)$ and the initial-conditions, $\mathbf{x}(0)$. For simplicity, let us assume that the nonlinear plant is *time-invariant*, i.e. $\mathbf{f}\{\mathbf{x}(t), \mathbf{u}(t), t\} = \mathbf{f}\{\mathbf{x}(t), \mathbf{u}(t)\}$, and the state-equation does not explicitly depend upon the time, t, in Eq. (9.51). Furthermore, let us assume for simplicity that the nonlinear function, $\mathbf{f}\{\mathbf{x}(t), \mathbf{u}(t)\}$, can be expressed in the following form:

$$\mathbf{f}\{\mathbf{x}(t), \mathbf{u}(t)\} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{F}\{\mathbf{x}(t)\} \tag{9.52}$$

where $\mathbf{F}\{\mathbf{x}(t)\}$ is a nonlinear vector function that depends *only* upon the state-vector, $\mathbf{x}(t)$. Equation (9.52) implies that there are no nonlinear terms involving the control input, $\mathbf{u}(t)$, in the state-equation. Such nonlinear plants are fairly common in applications such as robotics, spacecraft attitude control, bio-chemical dynamics, and economics [18]. Then a nonlinear *regulator* problem for *infinite-time* can be posed by finding an optimal control input, $\mathbf{u}(t)$, such that the following objective function is minimized:

$$J = \int_0^\infty [q(\mathbf{x}(t)) + \mathbf{u}^{\mathrm{T}}(t)\mathbf{R}\mathbf{u}(t)]\,dt \tag{9.53}$$

where $q\{\mathbf{x}(t)\}$ is a *positive semi-definite* function denoting the cost associated with the transient response, $\mathbf{x}(t)$, and $\mathbf{u}^{\mathrm{T}}(t)\mathbf{R}\mathbf{u}(t)$ is the *quadratic* cost associated with the control input, $\mathbf{u}(t)$, with the matrix, $\mathbf{R}$, being *symmetric* and *positive definite*. It can be shown by the *minimum principle* [7] that if *all* the *derivatives* of $\mathbf{F}\{\mathbf{x}(t)\}$ *with respect* to $\mathbf{x}(t)$ are *continuous* in the space formed by the elements of $\mathbf{x}(t)$, then the minimization of the objective function, J, given by Eq. (9.53) with respect to the control input vector, $\mathbf{u}(t)$, *subject to the constraint* that the state-vector, $\mathbf{x}(t)$, satisfies Eq. (9.51), is *equivalent* to the minimization of the following scalar function, called the *Hamiltonian*, with respect to the control input, $\mathbf{u}(t)$:

$$\mathcal{H} = q\{\mathbf{x}(t)\} + \mathbf{u}^{\mathrm{T}}(t)\mathbf{R}\mathbf{u}(t) + [dV\{\mathbf{x}(t)\}/d\mathbf{x}(t)][\mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{F}\{\mathbf{x}(t)\}] \tag{9.54}$$

where $V\{\mathbf{x}(t)\}$ is a *positive semi-definite* function with the property $V\{\mathbf{0}\} = 0$, called the *Lyapunov function*. Note that J, H, $q\{\mathbf{x}(t)\}$, and $V\{\mathbf{x}(t)\}$ are all *scalars*. (We have seen scalar functions of a vector in Chapter 6, such as $\mathbf{x}^{\mathrm{T}}(t)\mathbf{Q}\mathbf{x}(t)$ and $\mathbf{u}^{\mathrm{T}}(t)\mathbf{R}\mathbf{u}(t)$.) However, $dV\mathbf{x}(t)/d\mathbf{x}(t)$ is a *row-vector* (sometimes expressed as $dV\mathbf{x}(t)/d\mathbf{x}(t) = \mathbf{p}^T(t)$, where $\mathbf{p}(t)$ is called the *co-state vector* of the nonlinear system); the derivative of a scalar, $V\{\mathbf{x}(t)\}$, with respect to a vector, $\mathbf{x}(t)$, means the differentiation of the scalar, $V\{\mathbf{x}(t)\}$, by each element of the vector, $\mathbf{x}(t)$, and storing the result as the corresponding element of a vector of the same size as $\mathbf{x}(t)$ (see Appendix B). Note that the Hamiltonian, $\mathcal{H}$, includes the term $[\partial V\{\mathbf{x}(t)\}/\partial\mathbf{x}(t)][\mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{F}\{\mathbf{x}(t)\}]$, which can be seen as imposing a *penalty* on deviating from the state-equation, Eq. (9.51). Thus, the constraint of Eq. (9.51) is implicitly satisfied by minimizing the Hamiltonian, $\mathcal{H}$. The *necessary conditions* of optimal control can be expressed as follows [20]:

$$\partial\mathcal{H}/\partial\mathbf{u}(t) = \mathbf{0} \tag{9.55}$$

$$\mathcal{H}_{\min} = 0 \tag{9.56}$$

$$\mathbf{x}^{(1)}(t) = \partial\mathcal{H}/\partial\mathbf{p}(t)|_{\mathbf{u}(t)=\mathbf{u}^*(t)} \tag{9.57}$$

$$\mathbf{p}^{(1)}(t) = -\partial\mathcal{H}/\partial\mathbf{x}(t)|_{\mathbf{u}(t)=\mathbf{u}^*(t)} \tag{9.58}$$

where $\mathbf{p}(t) = [dV\{\mathbf{x}(t)\}/d\mathbf{x}(t)]^T$ (the co-state vector), and $\mathbf{u}^*(t)$ denotes the optimal control input (which minimizes $\mathcal{H}$). The result of Eq. (9.55) is the following expression for the *optimal* control input:

$$\mathbf{u}^*(t) = -\mathbf{R}^{-1}\mathbf{B}^T[\partial V\{\mathbf{x}(t)\}/\partial\mathbf{x}(t)]^T \tag{9.59}$$

It is clear from Eq. (9.59) that we must know the *Lyapunov function*, $V\{\mathbf{x}(t)\}$, (or the co-state vector $\mathbf{p}(t) = [dV\{\mathbf{x}(t)\}/d\mathbf{x}(t)]^T$) if we have any chance of finding the optimal control input, $\mathbf{u}^*(t)$. The Lyapunov function (or the co-state vector) depends upon the characteristics of the nonlinear plant. Mathematically, $\mathbf{p}(t)$, can be obtained from the coupled solution to the two-point boundary-value problem posed by Eqs. (9.57) and (9.58), which is not always possible to obtain analytically. However, since $\mathbf{p}(t)$ must also satisfy Eq. (9.56), which is an algebraic equation analogous to the *algebraic Riccati* equation for the linear problem, Eq. (9.56) gives a practical method of finding $\mathbf{p}(t) = [dV\{\mathbf{x}(t)\}/d\mathbf{x}(t)]^T$. Equation (9.56) is generally known as the *Hamilton–Bellman–Jacobi* equation (or, in short, the *Bellman* equation [21]) for the infinite-time problem. (For *finite-time* control, the Bellman equation becomes a *partial differential* equation analogous to the *matrix Riccati* equation for the linear problem.)

Once we have posed the optimal control problem by Eqs. (9.54)–(9.59), we can look at a workable solution procedure [22] which uses Eq. (9.56) to derive $dV\{\mathbf{x}(t)\}/d\mathbf{x}(t)$. However, in using such a procedure, we must specify a form for $V\{\mathbf{x}(t)\}$. Let us begin by expressing the transient response energy, $q\{\mathbf{x}(t)\}$, in the following form:

$$q\{\mathbf{x}(t)\} = (1/2)[\mathbf{x}^T(t)\ \{\mathbf{x}^2(t)\}^T \ldots \{\mathbf{x}^n(t)\}^T] \begin{bmatrix} \mathbf{Q_{11}} & \mathbf{Q_{12}} & \cdots & \mathbf{Q_{1n}} \\ \mathbf{Q_{21}} & \mathbf{Q_{22}} & \cdots & \mathbf{Q_{2n}} \\ \cdot & \cdot & \cdots & \cdot \\ \mathbf{Q_{n1}} & \mathbf{Q_{n2}} & \cdots & \mathbf{Q_{nn}} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{x}^2(t) \\ \cdot \\ \mathbf{x}^n(t) \end{bmatrix} \quad (9.60)$$

where $\mathbf{Q_{ij}} = \mathbf{Q_{ji}}$, and $\mathbf{x}^k(t)$ denotes a vector formed by raising each element of $\mathbf{x}(t)$ to the power k. This definition of $q\{\mathbf{x}(t)\}$ makes it positive semi-definite, as required, with a proper selection of $\mathbf{Q_{ij}}$. Note that $(2n-1)$ is the highest power of $\mathbf{x}(t)$ which appears in the nonlinear function, $\mathbf{F}\{\mathbf{x}(t)\}$, which can be written in the following form:

$$\mathbf{F}\{\mathbf{x}(t)\} = \sum_{k=2}^{(2n-1)} \mathbf{F_k}\{\mathbf{x}(t)\} \quad (9.61)$$

where $\mathbf{F}_k\{\mathbf{x}(t)\}$ denotes a *nonlinearity* of power k. Similarly, Eq. (9.60) can be re-written as

$$q\{\mathbf{x}(t)\} = \sum_{k=2}^{2n} q_k\{\mathbf{x}(t)\} \quad (9.62)$$

where

$$\begin{gathered} q_2\{\mathbf{x}(t)\} = \mathbf{x}^T(t)\mathbf{Q_{11}}\mathbf{x}(t); \quad q_3\{\mathbf{x}(t)\} = \mathbf{x}^T(t)\mathbf{Q_{12}}\mathbf{x}^2(t) + \{\mathbf{x}^2(t)\}^T\mathbf{Q_{21}}\mathbf{x}(t); \\ q_4\mathbf{x}(t) = \mathbf{x}^T(t)\mathbf{Q_{13}}\mathbf{x}^3(t) + \{\mathbf{x}^2(t)\}^T\mathbf{Q_{22}}\mathbf{x}^2(t) + \{\mathbf{x}^3(t)\}^T\mathbf{Q_{31}}\mathbf{x}(t); \\ \ldots \\ q_{2n}\{\mathbf{x}(t)\} = \{\mathbf{x}^n(t)\}^T\mathbf{Q_{nn}}\mathbf{x}^n(t) \end{gathered} \quad (9.63)$$

To determine a structure for the Lyapunov function, $V\{\mathbf{x}(t)\}$, that ensures its positive semi-definiteness, and satisfies the property $V\{\mathbf{0}\} = 0$, it is assumed that $V\{\mathbf{x}(t)\}$ has the *same form* as $q\{\mathbf{x}(t)\}$, i.e.

$$V\{\mathbf{x}(t)\} = (1/2)[\mathbf{x}^T(t)\ \{\mathbf{x}^2(t)\}^T \ldots \{\mathbf{x}^n(t)\}^T] \begin{bmatrix} \mathbf{P_{11}} & \mathbf{P_{12}} & \cdots & \mathbf{P_{1n}} \\ \mathbf{P_{21}} & \mathbf{P_{22}} & \cdots & \mathbf{P_{2n}} \\ \cdot & \cdot & \cdots & \cdot \\ \mathbf{P_{n1}} & \mathbf{P_{n2}} & \cdots & \mathbf{P_{nn}} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{x}^n(t) \\ \cdot \\ \mathbf{x}^2(t) \end{bmatrix} \quad (9.64)$$

where $\mathbf{P_{ij}} = \mathbf{P_{ji}}$. $V\{\mathbf{x}(t)\}$ can also be expressed as

$$V\{\mathbf{x}(t)\} = \sum_{k=2}^{2n} V_{\mathrm{k}}\{\mathbf{x}(t)\} \tag{9.65}$$

where

$$\begin{gathered}V_2\{\mathbf{x}(t)\} = \mathbf{x}^T(t)\mathbf{P_{11}}\mathbf{x}(t); \quad V_3\{\mathbf{x}(t)\} = \mathbf{x}^T(t)\mathbf{P_{12}}\mathbf{x}^2(t) + \{\mathbf{x}^2(t)\}^T\mathbf{P_{21}}\mathbf{x}(t);\\ V_4\{\mathbf{x}(t)\} = \mathbf{x}^T(t)\mathbf{P_{13}}\mathbf{x}^3(t) + \{\mathbf{x}^2(t)\}^T\mathbf{P_{22}}\mathbf{x}^2(t) + \{\mathbf{x}^3(t)\}^T\mathbf{P_{31}}\mathbf{x}(t);\\ \cdots \\ V_{2n}\{\mathbf{x}(t)\} = \{\mathbf{x}^n(t)\}^T\mathbf{P_{nn}}\mathbf{x}^n(t)\end{gathered} \tag{9.66}$$

Based on the expressions of $\mathbf{F}\{\mathbf{x}(t)\}$, $q\{\mathbf{x}(t)\}$, and $V\{\mathbf{x}(t)\}$ given by Eqs. (9.61), (9.62) and (9.65), the minimum value of the Hamiltonian can be written using Eq. (9.56) as

$$\mathcal{H}_{\min} = \sum_{k=2}^{2n} H_k = 0 \tag{9.67}$$

where

$$\begin{aligned}H_2 = {} & q_2\{\mathbf{x}(t)\} - [dV_2\{\mathbf{x}(t)\}/d\mathbf{x}(t)]\mathbf{BR^{-1}B}^T[dV_2\{\mathbf{x}(t)\}/d\mathbf{x}(t)]^T\\ & + [dV_2\{\mathbf{x}(t)\}/d\mathbf{x}(t)]\mathbf{A}\mathbf{x}(t) = 0\end{aligned} \tag{9.68}$$

$$\begin{aligned}H_3 = {} & q_3\{\mathbf{x}(t)\} - [dV_2\{\mathbf{x}(t)\}/d\mathbf{x}(t)]\mathbf{BR^{-1}B}^T[dV_3\{\mathbf{x}(t)\}/d\mathbf{x}(t)]^T\\ & - [dV_3\{\mathbf{x}(t)\}/d\mathbf{x}(t)]\mathbf{BR^{-1}B}^T[dV_2\{\mathbf{x}(t)\}/d\mathbf{x}(t)]^T + [dV_3\{\mathbf{x}(t)\}/d\mathbf{x}(t)]\mathbf{A}\mathbf{x}(t)\\ & + [dV_2\{\mathbf{x}(t)\}/d\mathbf{x}(t)]\mathbf{F_2}\{\mathbf{x}(t)\} = 0\end{aligned} \tag{9.69}$$

$$\begin{aligned}H_4 = {} & q_4\{\mathbf{x}(t)\} - [dV_2\{\mathbf{x}(t)\}/d\mathbf{x}(t)]\mathbf{BR^{-1}B}^T[dV_4\{\mathbf{x}(t)\}/d\mathbf{x}(t)]^T\\ & - [dV_3\{\mathbf{x}(t)\}/d\mathbf{x}(t)]\mathbf{BR^{-1}B}^T[dV_3\{\mathbf{x}(t)\}/d\mathbf{x}(t)]^T\\ & - [dV_4\{\mathbf{x}(t)\}/d\mathbf{x}(t)]\mathbf{BR^{-1}B}^T[dV_2\{\mathbf{x}(t)\}/d\mathbf{x}(t)]^T + [dV_4\mathbf{x}(t)/d\mathbf{x}(t)]\mathbf{A}\mathbf{x}(t)\\ & + [dV_3\mathbf{x}(t)/d\mathbf{x}(t)]\mathbf{F_2}\mathbf{x}(t) + [dV_2\mathbf{x}(t)/d\mathbf{x}(t)]\mathbf{F_3}\mathbf{x}(t) = 0\end{aligned} \tag{9.70}$$

$$\cdots$$

$$\begin{aligned}H_{2n} = {} & q_{2n}\{\mathbf{x}(t)\} - [dV_2\{\mathbf{x}(t)\}/d\mathbf{x}(t)]\mathbf{BR^{-1}B}^T[dV_{2n}\{\mathbf{x}(t)\}/d\mathbf{x}(t)]^T - \cdots\\ & - [dV_{2n-1}\{\mathbf{x}(t)\}/d\mathbf{x}(t)]\mathbf{BR^{-1}B}^T[dV_3\{\mathbf{x}(t)\}/d\mathbf{x}(t)]^T\\ & - [dV_{2n}\{\mathbf{x}(t)\}/d\mathbf{x}(t)]\mathbf{BR^{-1}B}^T[dV_2\{\mathbf{x}(t)\}/d\mathbf{x}(t)]^T + [dV_{2n}\{\mathbf{x}(t)\}/d\mathbf{x}(t)]\mathbf{A}\mathbf{x}(t)\\ & + [dV_{2n-1}\{\mathbf{x}(t)\}/d\mathbf{x}(t)]\mathbf{F_2}\{\mathbf{x}(t)\} + \cdots + [dV_2\{\mathbf{x}(t)\}/d\mathbf{x}(t)]\mathbf{F_{2n-1}}\{\mathbf{x}(t)\} = 0\end{aligned} \tag{9.71}$$

The Lyapunov parameters $\mathbf{P_{ij}}$ are determined by the following procedure [22]:

(a) Equation (9.68) – which has the *linear part* of the Hamiltonian, H_2 – can be expressed as the following *algebraic Riccati* equation:

$$\mathbf{A^T P_{11} + P_{11} A + Q_{11} - P_{11} B R^{-1} B^T P_{11} = 0} \tag{9.72}$$

Equation (9.72) is solved by standard procedures of Chapter 6 to get $\mathbf{P_{11}}$. Note that the *linear feedback* control input for the plant is given by $\mathbf{u}(t) = -\mathbf{R^{-1}B^T P_{11} x}(t)$.

(b) $\mathbf{P_{11}}$ is substituted into Eq. (9.69) to determine $\mathbf{P_{12}}(= \mathbf{P_{21}})$.

(c) $\mathbf{P_{11}}$ and $\mathbf{P_{12}}$ are substituted into Eq. (9.70) to determine $\mathbf{P_{13}}(= \mathbf{P_{31}})$ and $\mathbf{P_{22}}$.

(d) Continue successive substitution of known $\mathbf{P_{ij}}$ into Eqs. (9.70)–(9.71) until *all parameters*, including $\mathbf{P_{nn}}$, are found.

When all Lyapunov parameters are calculated, they are substituted into Eq. (9.64) to determine $V\{\mathbf{x}(t)\}$, which is then differentiated with respect to $\mathbf{x}(t)$ and substituted into Eq. (9.59) to determine the nonlinear optimal feedback control input, $\mathbf{u}^*(t)$.

Example 9.6

Consider the wing-rock dynamics of a fighter airplane described in Example 4.13 by nonlinear state-equations, Eq. (4.94), and programmed in the M-file *wrock.m* which is listed in Table 4.8. The state-space matrices (Eq. (9.52)) of the time-invariant, nonlinear plant are as follows:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -0.02013 & 0.0105 & 1 & -0.02822 & -0.1517 \\ 0 & 0 & -20.202 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0.0629 & 0 & -1.3214 & -0.2491 \end{bmatrix};$$

$$\mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 20.202 \\ 0 \\ 0 \end{bmatrix} \tag{9.73}$$

$$\mathbf{F}\{\mathbf{x}(t)\} = \left[0; \quad \{0.026x_2^3(t) - 0.1273x_1^2(t)x_2(t) + 0.5197x_1(t)x_2^2(t)\}; \; 0; \; 0; \; 0\right]^T$$

The *limit cycle* wing-rock motion when the airplane is excited by the initial condition $\mathbf{x}(0) = [0.2; 0; 0; 0; 0]^T$, was illustrated in Figures 4.17–4.19. Since the highest power of the elements of $\mathbf{x}(t)$ in the expression for $\mathbf{F}\{\mathbf{x}(t)\}$ in the present wing-rock model is 3, it follows that $n = 2$, and only Eqs. (9.68)–(9.70) are required for determining the unknown Lyapunov parameters $\mathbf{P_{11}}, \mathbf{P_{12}}(= \mathbf{P_{21}})$, and $\mathbf{P_{22}}$. Here $V_4\{\mathbf{x}(t)\} = (1/2)\{\mathbf{x}^2(t)\}^T \mathbf{P_{22}} \mathbf{x}^2(t)$, which implies $\mathbf{P_{13}} = \mathbf{P_{31}} = \mathbf{0}$. Following the nonlinear control derivation procedure of the previous section, the non-zero elements of the nonlinear control Lyapunov parameters are determined to be the following [23]:

$$\mathbf{P}_{12}(3,1) = [0.02013\mathbf{Q}_{12}(1,3) + \mathbf{Q}_{12}(1,1)]/[416.34\mathbf{P}_{11}(3,1) + 8.2162\mathbf{P}_{11}(3,3) + 0.4067] \quad (9.74)$$

$$\mathbf{P}_{12}(3,4) = -\mathbf{P}_{11}(3,4)\mathbf{P}_{12}(3,1)/\mathbf{P}_{11}(3,1) \quad (9.75)$$

$$\mathbf{P}_{12}(3,5) = \mathbf{P}_{11}(3,5)\mathbf{P}_{12}(3,4)/\mathbf{P}_{11}(3,4) \quad (9.76)$$

$$\mathbf{P}_{12}(2,1) = 49.6771[\mathbf{Q}_{12}(1,1) - 408.12\mathbf{P}_{11}(3,1)\mathbf{P}_{12}(3,1)] \quad (9.77)$$

$$\mathbf{P}_{12}(3,2) = [\mathbf{P}_{12}(3,1)(1 - 408.12\mathbf{P}_{11}(3,2) + \mathbf{P}_{12}(2,1)]/(408.12\mathbf{P}_{11}(3,1) + 0.02013) \quad (9.78)$$

$$\mathbf{P}_{12}(3,3) = [\mathbf{Q}_{12}(3,1) - 40.404\mathbf{P}_{12}(3,1)(20.202\mathbf{P}_{11}(3,3) + 1) - 0.02013\mathbf{P}_{12}(3,2)]/[1224.4\mathbf{P}_{11}(3,1) \quad (9.79)$$

$$\mathbf{P}_{22}(3,1) = -\mathbf{P}_{12}(3,1)^2/\mathbf{P}_{11}(3,1) \quad (9.80)$$

$$\mathbf{P}_{22}(3,2) = [0.026\mathbf{P}_{11}(3,3) - 816.24\mathbf{P}_{12}(3,2)\mathbf{P}_{12}(3,1)]/(816.24\mathbf{P}_{11}(3,1)) \quad (9.81)$$

$$\mathbf{P}_{22}(3,3) = -3\mathbf{P}_{12}(3,1)\mathbf{P}_{12}(3,3)/\mathbf{P}_{11}(3,1) \quad (9.82)$$

$$\mathbf{P}_{22}(3,4) = -\mathbf{P}_{12}(3,1)\mathbf{P}_{12}(3,4)/\mathbf{P}_{11}(3,1) \quad (9.83)$$

$$\mathbf{P}_{22}(3,5) = -\mathbf{P}_{12}(3,1)\mathbf{P}_{12}(3,5)/\mathbf{P}_{11}(3,1) \quad (9.84)$$

The nonlinear optimal control input is then obtained to be the following:

$$\begin{aligned}\mathbf{u}^*(t) = -20.202[&\mathbf{P}_{11}(3,1)x_1(t) + \mathbf{P}_{11}(3,2)x_2(t) + \mathbf{P}_{11}(3,3)x_3(t) + \mathbf{P}_{11}(3,4)x_4(t)\\ &+ \mathbf{P}_{11}(3,5)x_5(t) + \mathbf{P}_{12}(3,1)x_1^2(t) + \mathbf{P}_{12}(3,2)x_2^2(t) + 2\mathbf{P}_{12}(3,1)x_1(t)x_3(t)\\ &+ 2\mathbf{P}_{12}(3,2)x_2(t)x_3(t) + 3\mathbf{P}_{12}(3,3)x_3^2(t) + 2\mathbf{P}_{12}(3,4)x_3(t)x_4(t)\\ &+ 2\mathbf{P}_{12}(3,5)x_3(t)x_5(t) + \mathbf{P}_{12}(3,4)x_4^2(t) + \mathbf{P}_{12}(3,5)x_5^2(t)\\ &+ 2\mathbf{P}_{22}(3,1)x_1^2(t)x_3(t) + 2\mathbf{P}_{22}(3,2)x_2^2(t)x_3(t) + 2\mathbf{P}_{22}(3,3)x_3^3(t)\\ &+ 2\mathbf{P}_{22}(3,4)x_3(t)x_4^2(t) + 2\mathbf{P}_{22}(3,5)x_3(t)x_5^2(t)]\end{aligned} \quad (9.85)$$

The cost parameters $\mathbf{Q}_{11}$ and $\mathbf{Q}_{12}$ are selected such that for a *large initial condition*, such as $\mathbf{x}(0) = [1; 1; 0; 0; 0]^T$, the resulting closed-loop aileron response is limited to $\pm 35°$ and all the transients subside within 50 s. Furthermore, stability and performance robustness with respect to a 10 per cent variation in nonlinear aerodynamic and actuator parameters [23] must be ensured. The $\mathbf{Q}_{11}$, $\mathbf{Q}_{12}$ and $\mathbf{Q}_{22}$ matrices to achieve these specifications are the following:

$$\mathbf{Q}_{11} = \begin{bmatrix} 0.01 & 0 & 0 & 0 & 0 \\ 0 & 0.01 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0.001 & 0 \\ 0 & 0 & 0 & 0 & 0.001 \end{bmatrix}; \quad \mathbf{Q}_{12} = 0.005\mathbf{I}_{3\times 3};$$

$$\mathbf{Q}_{22} = 0.1\mathbf{I}_{2\times 2} \quad (9.86)$$

where $\mathbf{I_k}$ denotes a $(k \times k)$ identity matrix. The resulting solution, $\mathbf{P}_{11}$, of the algebraic Riccati equation (Eq. (9.73)) is the following [23]:

$$\mathbf{P}_{11} = \begin{bmatrix} 0.0499 & 0.1081 & 0.0037 & 0.0124 & -0.0076 \\ 0.1081 & 0.6927 & 0.0241 & 0.0952 & -0.0295 \\ 0.0037 & 0.0241 & 0.0213 & 0.0033 & -0.0010 \\ 0.0124 & 0.0952 & 0.0033 & 0.0189 & -0.0034 \\ -0.0076 & -0.0295 & -0.0010 & -0.0034 & 0.0057 \end{bmatrix} \tag{9.87}$$

The nonlinear optimal feedback closed-loop response for the initial condition $\mathbf{x}(0) = [1; 1; 0; 0; 0]^T$ is shown in Figure 9.5. For this initial condition, the linear feedback control input given by $\mathbf{u}(t) = -\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P}_{11}\mathbf{x}(t)$ fails to stabilize the plant [23]. In these figures, comparison is made for 10 per cent variation in the nonlinear aerodynamic parameters of $\mathbf{F}\{\mathbf{x}(t)\}$ from their nominal values given in Eq. (9.73). It is observed that the system's response is stable and within the specified performance limits for 10 per cent uncertainty in the aerodynamic parameters. Tewari [23] also shows robustness with respect to 10 per cent variation in the actuator model.

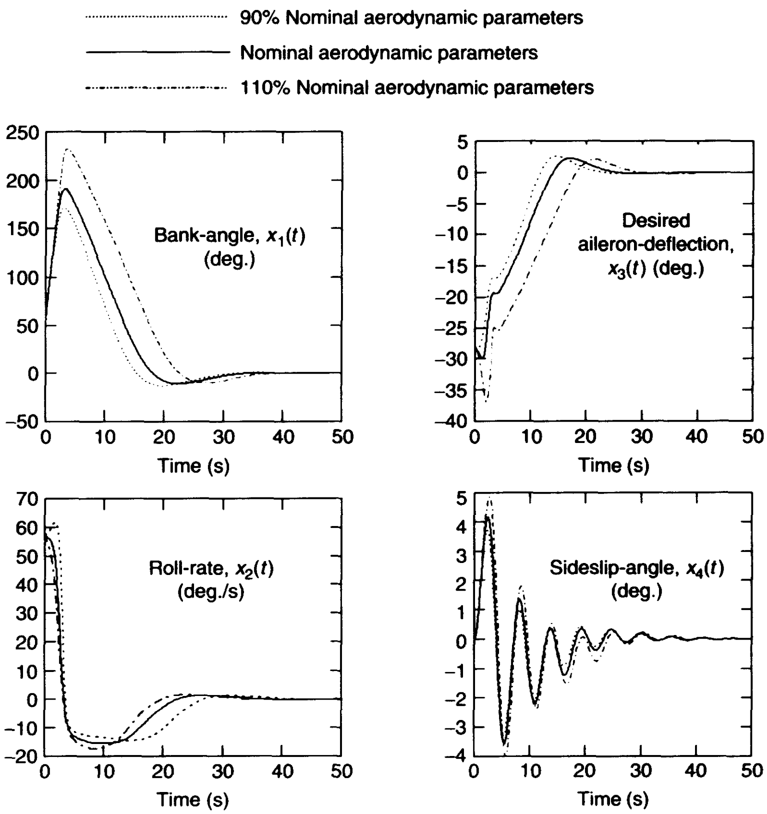

Figure 9.5 Closed-loop response of the nonlinear optimal feedback control system for the wing-rock suppression of a fighter airplane for a large initial condition