

Exercises

- 2.1. Consider an inverted pendulum of length L and mass m on a moving cart of mass, M (Figure 2.59). It is assumed that the cart moves on a frictionless, flat plane. The force, $f(t)$, applied to the cart is the input, while the output is the angular deflection, $\theta(t)$, of the pendulum from the vertical. The displacement of the cart from a fixed point is given by $x(t)$. The governing differential equations of the system are as follows:

$$(M + m)x^{(2)}(t) + mL\theta^{(2)}(t)\cos(\theta(t)) - mL[\theta^{(1)}(t)]^2\sin(\theta(t)) = f(t) \quad (2.202)$$

$$mx^{(2)}(t)\cos(\theta(t)) + mL\theta^{(2)}(t) - mg\sin(\theta(t)) = 0 \quad (2.203)$$

- What is the order of the system?
 - Eliminate $x(t)$ from Eqs. (2.202) and (2.203) to get a differential equation in terms of the output, $\theta(t)$, and input, $f(t)$.
 - Linearize the system about the equilibrium point, $\theta(t) = 0$.
 - Find the transfer function of the linearized system, $\Theta(s)/F(s)$, where $\Theta(s) = \mathcal{L}\theta(t)$ and $F(s) = \mathcal{L}f(t)$.
 - What are the poles of the linearized system? Is the linearized system stable?
- 2.2. Certain unforced physical systems obey the following governing differential equation, called *van der Pol* equation:

$$x^{(2)}(t) + a[x(t)^2 - 1]x^{(1)}(t) + bx(t) = 0 \quad (2.204)$$

where $x(t)$ is the output variable, and a and b are positive constants. Can you linearize such a system about an equilibrium point?

- 2.3. The governing equations of an electrical network (Figure 2.60(a)) are as follows:

$$i_1(t) = f(v_1(t)) \quad (2.205)$$

$$v_1^{(1)}(t) = [v_2(t) - v_1(t)]/(R_2C_1) - i_1(t)/C_1 \quad (2.206)$$

$$v_2^{(1)}(t) = [v_1(t) - v_2(t)]/(R_2C_2) - i_2(t)/C_2 \quad (2.207)$$

$$i_2^{(1)}(t) = -v_2(t)/L \quad (2.208)$$

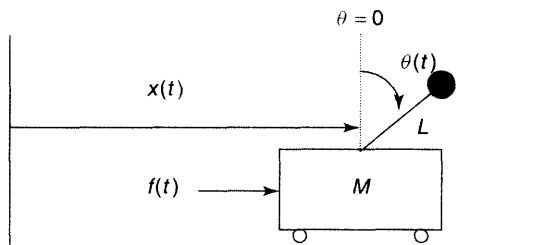


Figure 2.59 Inverted pendulum on a moving cart

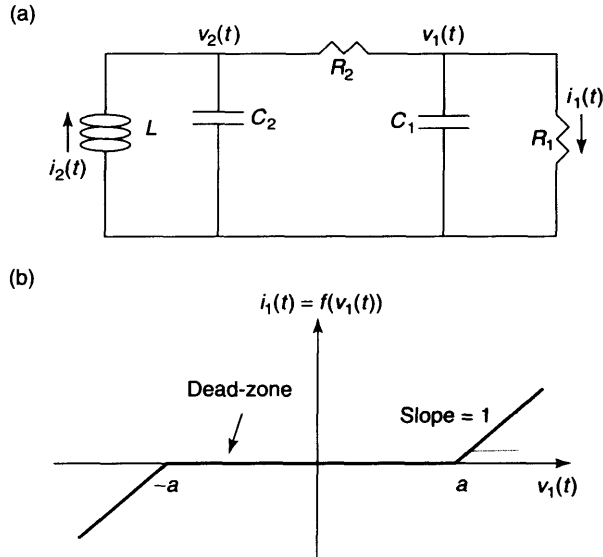


Figure 2.60 (a) Electrical network for Exercise 2.3; (b) Nonlinear current, $i_1(t)$, with a dead-zone in the voltage band $-a < v_1(t) < a$

where $i_1(t)$ is the current flowing through a nonlinear resistance, R_1 , $i_2(t)$ is the current flowing through the inductor, L , and $v_1(t)$ and $v_2(t)$ are the voltages across the capacitors, C_1 and C_2 , respectively. The function $f(v_1(t))$ is shown in Figure 2.60(b). Such a nonlinear function is said to have a *dead-zone* in the voltage band, $-a < v_1(t) < a$.

- Find the equilibrium points of the system, and linearize the system about each of them.
- Derive the transfer function, $G(s)$, between the current, $i_1(t)$, as the output and the current, $i_2(t)$, as the input, of the system linearized about each equilibrium point.
- Investigate the stability of the system about each equilibrium point.
- What is the step response of the linearized system when the current $i_1(t)$ is in the dead-zone?

2.4. Linearize the governing differential equation of a satellite in an orbit around a planet (Eq. 2.16) about the equilibrium point denoted by the circular orbit, $r(t) = C$. What are the natural frequency and damping-ratio of the linearized orbital dynamics? Is the orbital system stable about the equilibrium point? Find the response of the deviation of the satellite from the equilibrium point denoted by the circular orbit, if the input radial acceleration is 1000 m/s^2 applied as a step input at $t = 0$, i.e. $u(t) = 1000u_s(t)$. What are the maximum overshoot, settling time, and steady-state value of the response? (Assume that the satellite is in a circular orbit of radius, $C = 3.0 \times 10^8 \text{ m}$, around Mars, with an angular momentum, $h = 3.0 \times 10^{12} \text{ m}^2/\text{s}$. For Mars, $k^2 = 4.27 \times 10^{13} \text{ m}^3/\text{s}^2$.)

2.5. Repeat Exercise 2.4 for a satellite in orbit around Jupiter, with $C = 4.0 \times 10^9 \text{ m}$, $h = 4.8 \times 10^{13} \text{ m}^2/\text{s}$, and $k^2 = 1.27 \times 10^{17} \text{ m}^3/\text{s}^2$.

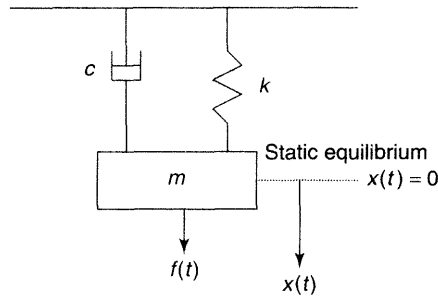


Figure 2.61 Spring-mass-damper system with input, $f(t)$, and output, $x(t)$

- 2.6. Consider the spring-mass-damper system shown in Figure 2.61, with mass, m , spring stiffness, k , and damping coefficient, c . The deflection, $x(t)$, of the mass is measured from its *static equilibrium position* (given by $x(t) = 0$), and is the system's output. A force, $f(t)$, applied to the mass is the input. The governing differential equation of the system is

$$mx^{(2)}(t) + cx^{(1)}(t) + kx(t) = f(t) \quad (2.209)$$

- (a) Derive an expression for the system's frequency response.
 - (b) What are the expressions for the natural frequency, ω_n , and damping-ratio, ζ , of the system?
 - (c) Assuming zero initial conditions, derive the step response and impulse response of the system. Plot the step response for $\zeta = 0.1$ and $\zeta = 2.0$. Calculate the maximum percentage overshoot, peak-time and settling-time in both the values of ζ .
 - (d) If $c = 0$, is the system stable?
- 2.7. Calculate the impulse response of the following transfer functions:
- (a) $(s + 1)/[(s + 0.01)(s + 0.3)]$
 - (b) $(s^2 + 3s + 1)/(s^3 + 2s^2 + 7s + 10)$
 - (c) $10s/(s^2 + 2s - 1)$
- 2.8. Plot the step response of the following transfer functions, and calculate the maximum overshoot, settling time, and the steady-state output:
- (a) $s/(s + 5)$
 - (b) $(s^2 + 1)/(s^2 + 3s + 2)$
 - (c) $10(s^3 + 2s^2 + 4s + 1)/(s^4 + s^3 + 10s^2 + 5s + 10)$
- 2.9. Calculate and plot the response of a system with transfer function, $G(s) = 100(s^2 + 2)/(s^3 + s^2 + 5s + 1)$ to the input, $u(t) = 10e^{-t} \sin(5t)$, assuming zero initial conditions. What is the magnitude of the maximum overshoot? What is the steady-state value of the output?

- 2.10. For the linearized satellite orbital dynamics of Exercise 2.4, plot the deviation from the circular orbit if the input is given by $u(t) = 5000[\sin(0.1t) + \cos(0.2t)]$, assuming zero initial conditions. What is the magnitude of the maximum deviation from the circular orbit?
- 2.11. Plot the response of the spring-mass-damper system of Exercise 2.5, if $m = 1$ kg, $k = 10$ N/m, $c = 0.1$ Ns/m, and $f(t) = 10e^{-2t} \cos(10t)$, assuming zero initial conditions.
- 2.12. The transfer function of a control system is given by

$$Y(s)/U(s) = G(s) = 20(s^3 + 4s^2 + 5s + 2)/(s^5 + 15s^4 + 30s^3) \quad (2.210)$$

- (a) Investigate the stability of the system.
 - (b) Plot the response to a unit ramp function, $r(t)$, for zero initial conditions.
 - (c) Draw a Bode plot of the system.
- 2.13. For a second-order linear system, prove that the step response goes to infinity in the limit $t \rightarrow \infty$ if the system has *repeated poles with real part equal to zero*.
- 2.14. Consider the control system shown in Figure 2.32, where the plant's transfer function is $G(s) = 1/[(s + 0.1)(s + 0.2)]$. It is intended to use a PID compensator for controlling this plant.
- (a) What are the values of the PID compensator constants, K_P , K_I , and K_D , for achieving a zero steady-state error with a closed-loop pole at $s = -1$, and two complex conjugate poles with damping ratio, $\zeta = 0.707$, and natural frequency, $\omega_n = 1$ rad/s?
 - (b) Derive the closed-loop transfer function, $Y(s)/Y_d(s)$, with the compensator in part (a), and compute the closed-loop system step response. What is the maximum percentage overshoot, settling time, and steady-state error of the closed-loop step response?
 - (c) Plot the locus of the closed-loop poles as K_I varies from 0 to 10, with K_D and K_P remaining constant at the values calculated in part (a). What are the values of K_I for which the closed-loop system is stable?
 - (d) Draw a Bode plot of the closed-loop system of part (a), and determine the gain and phase margins, and the respective crossover frequencies.
- 2.15. For the control system shown in Figure 2.32, it is desired to track an object moving with a *constant acceleration* in m/s^2 given by $y_d(t) = 5t^2 \cdot u_s(t)$, by moving an antenna whose transfer function is $G(s) = 20/[s(s + 100)]$. Find a controller transfer function, $H(s)$, such that the steady-state tracking error is less than 0.001 m/s^2 , with a settling time of about 0.1 seconds. Plot the closed-loop error response as a function of time. Determine the gain and phase margins, and the respective crossover frequencies of the closed-loop system. What are the maximum overshoot, settling time, and steady-state error of the closed-loop system if the desired output is a unit step function?

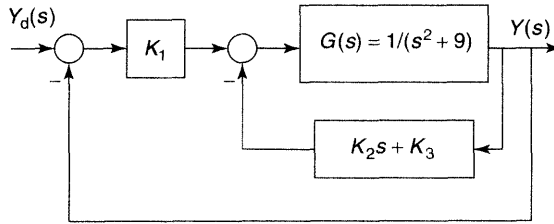


Figure 2.62 Two loop control system for Exercise 2.16

2.16. (a) Derive the transfer function, $Y(s)/Y_d(s)$, of the closed-loop system shown in Figure 2.62.

(b) What is the type of this system?

(c) Find the values of K_1 , K_2 , and K_3 such that the closed-loop transfer function becomes

$$Y(s)/Y_d(s) = (s + 10)/(s + 2)(s^2 + 4s + 5) \quad (2.211)$$

(d) Determine the step response of the system in part (c).

(e) Using the convolution integral, determine the response of the system in part (c) to a rectangular pulse (Figure 2.13) of magnitude, $f_o = 1$ and duration, $T = 10$ s.

(f) Determine the gain and phase margins, and the respective crossover frequencies of the system in part (c).

2.17. For the control system in Exercise 2.16, determine the gain and phase margins if $K_1 = 10$, $K_2 = 25$, and $K_3 = 150$.

2.18. For the control system shown in Figure 2.63:

(a) Derive the closed-loop transfer function, $Y(s)/Y_d(s)$. If this control system is to be expressed as a *negative feedback* connection of two systems, $G(s)$ and $H(s)$, as shown in Figure 2.35, identify the *plant* transfer function, $G(s)$, and the *controller* transfer function, $H(s)$.

(b) What is the type of this system?

(c) Is the plant, $G(s)$, stable?

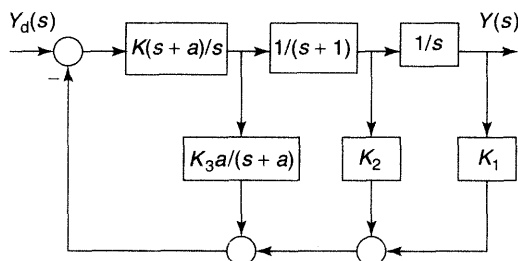


Figure 2.63 Control system for Exercise 2.18

- (d) Plot the locus of closed-loop poles if $K = 10$, $a = 2$, $K_2 = 25$, $K_3 = 150$, and K_1 is varied from 1 to 500.
- (e) What is the value of K_1 for which the closed-loop system has a dominant pair of complex poles with natural frequency, $\omega_n = 1.83$ rad/s and damping-ratio, $\zeta = 0.753$?
- (f) Plot the step response of the closed-loop system for the value of K_1 desired in part (e). What is the resulting maximum percentage overshoot, settling-time and the steady-state error?
- (g) Draw the Bode plots of the plant, $G(s)$, the controller, $H(s)$, and the closed-loop system, and find the gain and phase margins of the control system with K_1 of part (e).
- 2.19. For the electrical system of Example 2.8, consider a feedback control system of Figure 2.32 using a constant controller transfer function $H(s) = K$.
- (a) Draw a root-locus of the closed-loop poles as K is varied. What is the range of K for which the closed-loop system is stable?
- (b) Find the value of K for which the closed-loop system has a damping ratio, $\zeta = 0.707$. Plot the step response of the closed-loop system, and find the maximum overshoot, settling time, and steady-state error.
- 2.20. Re-design the controller, $H(s)$, in Exercise 2.19 to make the steady-state error zero, and the damping ratio, $\zeta = 0.707$. Plot the step response of the closed-loop system, and find the maximum overshoot and settling time.
- 2.21. For the closed-loop system shown in Figure 2.32, the plant's transfer function is $G(s) = 150\,000/(s^3 + 110s^2 + 1000s)$. It is desired to use a *lag compensator* to control this plant.
- (a) Draw a Bode plot of the plant, $G(s)$, and find the gain and phase margins of the plant.
- (b) Draw a Bode plot of the *appropriate* lag compensator, $H(s)$, for $\omega_o = 1$ rad/s and $\alpha = 10$. What is the frequency at which the phase of $H(s)$ is the minimum? What is the change in the gain of $H(s)$ between 0 and 10 rad/s?
- (c) Compare the steady-state error, maximum overshoot, and settling time of the plant with those of the closed-loop system with the controller parameters of part (b), if the desired output is a unit step function.
- (d) Draw a Bode plot of the closed-loop system with the lag compensator of part (b). Find the gain and phase margins of the closed-loop system.
- (e) Repeat parts (b)–(d) with $\omega_o = 1$ rad/s and $\alpha = 1.1$.
- 2.22. For the closed-loop system shown in Figure 2.32, the plant's transfer function is $G(s) = 1000/(s^2 + 2s + 5)$. It is desired to use a *lead compensator* to control this plant.
- (a) Draw a Bode plot of the plant, $G(s)$, and find the gain and phase margins of the plant.
- (b) Draw a Bode plot of the lead compensator, $H(s)$, for $\omega_o = 40$ rad/s and $\alpha = 0.01$. What is the frequency at which the phase of $H(s)$ becomes a maximum? What is the total change in gain over the entire frequency range?

- (c) Compare the steady-state error, maximum overshoot, and settling time of the plant with those of the closed-loop system if the desired output is a unit step function.
- (d) Draw a Bode plot of the closed-loop system with the lead compensator of part (b). Find the gain and phase margins of the closed-loop system.
- (e) Repeat parts (b)–(d) with $\omega_o = 40$ rad/s and $\alpha = 0.45$.

2.23. A plant's transfer function is $G(s) = 10^6/(s^2 + 0.99s - 0.01)$. It is desired to use a *lead-lag compensator* in the control system of Figure 2.32 to control this plant.

- (a) Draw a Bode plot of the plant, $G(s)$, and find the gain and phase margins of the plant.
- (b) Draw a Bode plot of the lead-lag compensator, $H(s)$, for $\omega_1 = 0.1$ rad/s, $\omega_2 = 1000$ rad/s and $\alpha = 5$. What are the frequencies corresponding to minimum and maximum phase of $H(s)$?
- (c) Compare the steady-state error, maximum overshoot, and settling time of the plant with those of the closed-loop system if the desired output is a unit step function.
- (d) Draw a Bode plot of the closed-loop system with the lead-lag compensator of part (b). Find the gain and phase margins of the closed-loop system.
- (e) Keeping the values of ω_1 and ω_2 the same as in part (b), what is the value of α for which the maximum overshoot of the closed-loop step response is 30 percent? What is the corresponding settling time? Calculate the new closed-loop gain and phase margins for this value of α .

2.24. Suppose the closed-loop system shown in Figure 2.32 has plant's transfer function, $G(s) = (s + 1)/(s^2 + 2s - 3)$ and controller's transfer function, $H(s) = K/s$. Such a controller is called an *integral compensator*, since it is obtained from the PID compensator by setting $K_I = K$, and $K_P = K_D = 0$.

- (a) Is the plant alone stable?
- (b) Derive the closed-loop transfer function, $Y(s)/Y_d(s)$.
- (c) From a root-locus determine the range of variation of K for which the closed-loop system stable.
- (d) Derive the step response of the closed-loop system for $K = 50$. What is the maximum percentage overshoot, peak-time, settling-time and the steady-state error of the system?
- (e) What is the steady-state error of the closed-loop system with $K = 50$ if $y_d(t) = t \cdot u_s(t)$?
- (f) Determine the gain and phase margins, and the respective crossover frequencies of the system for $K = 50$.
- (g) Repeat (d)–(f) for $K = 15$.

2.25. For the aircraft in Example 2.10, it is desired to increase the damping in *phugoid mode* by using a *lead compensator*, with the plant's transfer function is $G(s) = v(s)/\delta(s)$.

- (a) Find the gain and phase margins for the plant, $G(s) = v(s)/\delta(s)$.
- (b) Find the values of the lead compensator parameters, ω_o and α , such that the closed-loop system has a gain margin greater than 10 dB and a phase margin greater than 130° .
- (c) Compare the settling time and maximum overshoot of the plant with those of the closed-loop system with the lead compensator designed in part (b), if the desired output is a unit step function, $u_s(t)$.

2.26. For the aircraft in Example 2.10, it is desired to reduce the *steady-state error* in the angle of attack, $\alpha(s)$, (which is largely influenced by the *short-period mode*) by using a *lag compensator* as a controller in the closed-loop configuration of Figure 2.32, where the plant's transfer function is $G(s) = \alpha(s)/\delta(s)$. From Figure 2.27, it is clear that the *phugoid mode* does not appreciably affect, $\alpha(s)/\delta(s)$, implying that the denominator quadratic corresponding to the phugoid mode, $(s^2 + 0.005s + 0.006)$, gets *approximately* canceled by the numerator quadratic $(s^2 + 0.0065s + 0.006)$ in the expression for $\alpha(s)/\delta(s)$ given by Eq. (2.87). Thus, we can write the *approximate* plant transfer function, $\alpha(s)/\delta(s)$, as follows:

$$\alpha(s)/\delta(s) \approx -0.02(s + 80)/(s^2 + s + 1.4) \quad (2.212)$$

- (a) Compare the Bode plot of the approximate $\alpha(s)/\delta(s)$ given by Eq. (2.212) with that shown in Figure 2.27 for the exact $\alpha(s)/\delta(s)$.
- (b) Compare the gain and phase margins for the approximate and exact plant transfer function, $G(s) = \alpha(s)/\delta(s)$.
- (c) Find the values of the appropriate lag compensator parameters, ω_o and α , such that the closed-loop system has a gain margin greater than 11 dB and a phase margin greater than 130° .
- (d) Compare the settling time and maximum overshoot of the plant with those of the closed-loop system with the lag compensator designed in part (b), if the desired output is a unit step function, $u_s(t)$.

2.27. In an aircraft, the *actuator* of a *control surface* – such as the *elevator* in Example 2.10 – takes some (*non-zero*) time to achieve the *desired* control-surface deflection angle, $\delta(t)$. The simplest model for such an actuator is given by a *first-order* transfer function, $\delta(s)/\delta_d(s) = 1/(Ts + 1)$, where T is a time-constant, and $\delta_d(s)$ is the desired deflection angle.

- (a) If $T = 0.02$ second, find the step response of $\delta(s)/\delta_d(s) = 1/(Ts + 1)$ (i.e. $\delta(t)$ when $\delta_d(t) = u_s(t)$). What is the settling time and the steady-state error?
- (b) Make a Bode plot of the actuator transfer function for $T = 0.02$ second. What are the DC gain, and gain and phase margins of the actuator?
- (c) For the actuator of parts (a) and (b), plot the actual control-surface deflection, $\delta(t)$, when the desired deflection, $\delta_d(t)$, is a *rectangular pulse* of width 0.5 second and height 0.05 radian, applied at $t = 0$. (Such a pulse *elevator* input is normally applied by the pilot to excite the *phugoid mode*.)

- (d) To account for the presence of the actuator, the longitudinal transfer functions of Example 2.10, are multiplied by the elevator actuator transfer function, $\delta(s)/\delta_d(s) = 1/(Ts + 1)$, resulting in the transfer functions, $v(s)/\delta_d(s)$, $\alpha(s)/\delta_d(s)$, and $\theta(s)/\delta_d(s)$. If $T = 0.02$ second, compare the Bode plots of $v(s)/\delta_d(s)$, $\alpha(s)/\delta_d(s)$, and $\theta(s)/\delta_d(s)$, with those of $v(s)/\delta(s)$, $\alpha(s)/\delta(s)$, and $\theta(s)/\delta(s)$ (shown in Figures 2.26–2.28), respectively. Is there any difference in the corresponding gain and phase margins?
- (e) Plot the step response of $\theta(s)/\delta_d(s)$, and compare it with that of $\theta(s)/\delta(s)$ shown in Figure 2.29. Is there a difference in the two step responses?
- (f) Repeat the design of a lead compensator for controlling the phugoid mode carried out in Exercise 2.25 with a plant transfer function, $G(s) = v(s)/\delta_d(s)$, instead of $G(s) = v(s)/\delta(s)$.

2.28. For controlling the deviation of a satellite from a circular orbit (Exercise 2.4), it is desired to use a *derivative* compensator, $H(s) = Ks$, in the *feedback* arrangement of Figure 2.35, such that the plant's transfer function is $G(s)$, and the feedback controller transfer function is $H(s)$. It is required that the closed-loop response to a *step input* of magnitude 1000 m/s^2 (i.e. $u(t) = 1000u_s(t)$) should have a maximum overshoot of 2 m and a settling time less than 13 seconds. Find the value of K which achieves this, and plot the closed-loop step response. What are the maximum overshoot and the settling time? Plot the *output*, $z(t)$, of the compensator, $H(s)$. What is the maximum magnitude of the controller output, $z(t)$? Draw the Bode plots of the plant, the compensator, and the closed-loop system, and compare the gain and phase margins of the plant with those of the closed-loop system.

2.29. Find the poles and zeros, and analyze the stability of multivariable systems with the following transfer matrices:

(a) $\mathbf{G}(s) = [(s + 1)/(s + 3)s/(s^2 - 7s + 1)]$

(b) $\mathbf{G}(s) = \begin{bmatrix} 1/(s + 1) & 0 \\ -1/(s + 2) & 2(s + 1)/(s^2 - 1) \end{bmatrix}$

(c) $\mathbf{G}(s) = \begin{bmatrix} (s + 4)/(s^2 + 3s + 2) & s/(s + 2) \\ 1/(s + 1) & 1/s \\ 1/s & 1/(s + 4) \end{bmatrix}$

2.30. Consider a multivariable system of Figure 2.32, with plant transfer matrix, $\mathbf{G}(s)$, and controller transfer matrix, $\mathbf{H}(s)$, given by

$$\mathbf{G}(s) = (1/s) \begin{bmatrix} 10 & 9 \\ 9 & 8 \end{bmatrix}; \quad \mathbf{H}(s) = \begin{bmatrix} K & 1 \\ 1 & K \end{bmatrix} \quad (2.213)$$

- (a) Derive the characteristic equation of the system.
- (b) Plot the root-locus as K varies from 0.01 to 10. For what range of variation of K is the system stable?