Robustness of digital, optimally compensated (DLQG) systems can be ensured in a manner similar to the analog multivariable systems presented in Section 7.6 using loop-transfer recovery (LTR), with the difference that the singular values are obtained for digital systems rather than analog systems. MATLAB's *Robust Control Toolbox* provides the command *dsigma* for computing singular values of a digital system. The command *dsigma* is used in precisely the same manner as the command *sigma* for analog systems, with the input arguments being the digital state-space coefficient matrices (or the numerator and denominator polynomials of single-input, single-output pulse transfer function). In a manner similar to the digital magnitude plot obtained by *dbode*, the command *dsigma* computes the singular values only upto the Nyquist frequency, $\pi/T$. As noted earlier, a digital system is unable to respond to frequencies above the Nyquist frequency, and thus has an inherent robustness with respect to high-frequency noise. As an exercise, you should carry out loop-transfer recovery for the plant of Example 2.22 using the techniques of Section 7.6 and the command *dsigma*.

Whereas digital control systems are quite robust with respect to high-frequency noise, their implementation using *integer*, or *binary* (rather than *real*) arithmetic results in some robustness problems that are not encountered in analog systems. The effects of *round-off, quantization*, and *overflow* when handling integer arithmetic on a digital computer appear as *additional noise* in digital systems, and robustness of a digital control system to such noise–referred to as *finite-word length effects* becomes important. A discussion of these topics is beyond the scope of this book, and you may refer to a textbook on digital control [2, 4] for this purpose.

## Exercises

8.1. Find the $z$-transform, $F(z)$, of each of the following digital functions:

(a) $f(kT) = e^{-akT}u_s(kT)$

(b) $f(kT) = (kT)^2 e^{-akT}u_s(kT)$

(c) $f(kT) = u_s(kT)\cos(akT)$

8.2. Find the inverse $z$-transform, $f(kT)$, of each of the following functions:

(a) $F(z) = z/(z^2 + 0.99)$

(b) $F(z) = z(z+1)/(z^3 - 2z^2 - 5z - 1)$

(c) $F(z) = z(z+1.1)(z-0.9)/(z^4 + 5z^3 - 2.2z^2 - 3)$

(d) $F(z) = (z+1)/[(z^2 - 1.14z + 0.65)(z - 0.9)]$

8.3. Find the pulse transfer function, $G(z)$, for each of the following sampled-data analog systems with transfer function, $G(s)$, and sampling interval, $T = 0.2$ second:

(a) $G(s) = (s+2)/[s(s+1)]$

(b) $G(s) = s/(s^2 - 2s + 6)$

(c) $G(s) = 100/(s^3 - 4s + 25)$

(d) $G(s) = (s+1)(s-2)/[(s+10)(s^3 + 5s^2 + 12s + 55)]$

8.4. Find the pulse transfer function of the closed-loop system consisting of an A/D converter with z.o.h and an analog plant of transfer function, $G(s)$, in the configuration shown in Figure 8.4, for each $G(s)$ given in Exercise 8.3.

8.5. Plot the digital Bode diagrams for each of the sampled-data analog systems in Exercise 8.3, and analyze their stability. What is the range of the sampling interval, $T$, for which each of the systems are stable?

8.6. Plot the digital Bode diagrams for each of the closed-loop digital systems in Exercise 8.4, and analyze their stability. What is the range of the sampling interval, $T$, for which each of the systems are stable?

8.7. Compute and plot the digital step responses of the sampled-data analog systems in Exercise 8.3.

8.8. Compute and plot the digital step responses of the closed-loop digital systems in Exercise 8.4.

8.9. Find the steady-state value of the output of each of the digital systems in Exercises 8.3 and 8.4 if the input is a *unit ramp function*, $tu_s(t)$.

8.10. Find the steady-state value of the output of each of the digital systems in Exercises 8.3 and 8.4 if the input is a *unit parabolic function*, $t^2u_s(t)$.

8.11. A robotic welding machine has the transfer function, $G(s) = 10/(s^3 + 9s^2 + 11\,000)$. A computer based closed-loop digital controller is to be designed for controlling the robot with the configuration shown in Figure 8.8. Let the computer act as an ideal sampler and z.o.h with a sampling interval of $T = 0.1$ second, and the digital controller be described by a *constant* pulse transfer function, $H(z) = K$. Find the range of $K$ for which the closed-loop system is stable.

8.12. The pitch dynamics of a satellite launch vehicle is described by the analog transfer function, $G(s) = 1/(s^2 - 0.03)$. A digital control system is to be designed for controlling this vehicle with a z.o.h, a sampling interval $T = 0.02$ second, and a constant controller pulse transfer function, $H(z) = K$. Find the range of $K$ for which the closed-loop system is stable. What is the value of $K$ for which the closed-loop step response settles in less than five seconds with a maximum overshoot less than 10 percent?

8.13. For the hard-disk read/write head of Example 2.23 with the analog transfer function given by Eq. (2.165), it is desired to design a computer based closed-loop digital system shown in Figure 8.8. The computer is modeled as an A/D converter with z.o.h and a sampling interval of $T = 0.05$ second, and the digital controller is a *lag-compensator* given by the pulse transfer function, $H(z) = K(z - z_o)/(z - z_p)$, where $K$ is a constant, $z_o = (2/T - \alpha\omega_o)/(2/T + \alpha\omega_o)$, and $z_p = (2/T - \omega_o)/(2/T + \omega_o)$, with $\alpha > 1$. Find the values of $K$, $\alpha$, and $\omega_o$ such that the closed-loop system has a maximum overshoot less than 20 percent and a steady-state error less than 1 percent if the desired output is a unit step function.

8.14. For the roll dynamics of a fighter aircraft described in Example 2.24 with the analog transfer function given by Eq. (2.181), it is desired to design a computer based closed-loop digital system shown in Figure 8.8. The computer is modeled as an A/D converter with z.o.h and a sampling interval of $T = 0.2$ second, and the digital controller is a *lead-compensator* given by the pulse transfer function, $H(z) = K(z - z_o)/(z - z_p)$, where $K$ is a constant, $z_o = (2/T - \alpha\omega_o)/(2/T + \alpha\omega_o)$, and $z_p = (2/T - \omega_o)/(2/T + \omega_o)$, with $\alpha < 1$. Find the values of $K$, $\alpha$, and $\omega_o$ such that the closed-loop system has a maximum overshoot less than 5 percent with a settling time less than two seconds and a zero steady-state error, if the desired output is a unit step function.

8.15. Derive a digital state-space representation for each of the sampled-data analog systems in Exercise 8.3.

8.16. Derive a digital state-space representation for each of the closed-loop digital systems in Exercise 8.4.

8.17. Find a state-space representation for each of the digital systems described by the following difference equations, with $y(k)$ as the output and $u(k)$ as the input at the $k$th sampling instant:

(a) $y(k+3) = 2y(k+2) - 1.3y(k+1) - 0.8y(k) + 0.2u(k)$

(b) $y(k+2) = 4y(k+1) - 3y(k) + 2u(k+1) - u(k)$

(c) $y(k+4) = 21y(k+3) - 15y(k+2) - y(k) + 3u(k+3) - u(k+1) + 2.2u(k)$

8.18. Find the controller companion form, the observer companion form, and the Jordan canonical form state-space representations of the digital systems in Exercise 8.17.

8.19. A multivariable digital system is described by the following difference equations:

$$y_1(k+2) + y_1(k+1) - y_2(k) = u_1(k) + u_2(k)$$

$$y_2(k+1) + y_2(k) - y_1(k) = u_1(k)$$

where $u_1(k)$ and $u_2(k)$ are the inputs and $y_1(k)$ and $y_2(k)$ are the outputs at the $k$th sampling instant. Derive a state-space representation of the system.

8.20. For the distillation column whose analog state-space representation is given in Exercises 5.4 and 5.15, derive a digital state-space representation with z.o.h and a sampling interval of $T = 0.2$ second.

8.21. For the aircraft lateral dynamics described by Eq. (4.97) in Exercise 4.3, derive a digital state-space representation with z.o.h and a sampling interval of $T = 0.2$ second.

8.22. Repeat Exercises 8.20 and 8.21 with a first-order hold. Compare the respective $z$-plane pole locations with those in Exercises 8.20 and 8.21.

8.23. Repeat Exercises 8.20 and 8.21 with a bilinear transformation. Compare the respective $z$-plane pole locations with those in Exercises 8.20 and 8.21.

8.24. For the distillation column with digital state-space representation derived in Exercise 8.20, design a full-state feedback regulator to place closed-loop poles at $z_{1,2} = 0.9 \pm 0.1i$, $z_3 = 0.37$, and $z_4 = 0.015$. Find the initial response of the regulated system to the initial condition $\mathbf{x}(0) = [1; 0; 0; 0]^T$.

8.25. For the aircraft lateral dynamics with digital state-space representation derived in Exercise 8.21, design a full-state feedback regulator to place closed-loop poles at $z_{1,2} = 0.9 \pm 0.1i$, $z_3 = 0.92$, and $z_4 = 0.22$. Find the initial response of the regulated system to the initial condition $\mathbf{x}(0) = [0.5; 0; 0; 0]^T$.

8.26. Using the MATLAB (CST) command *place*, design a full-order, two-output, current observer for the digitized distillation column of Exercise 8.20 such that the observer poles are placed at $z_{1,2} = 0.8 \pm 0.2i$, $z_3 = 0.3$, and $z_4 = 0.015$. Combine the observer with the regulator designed in Exercise 8.24 to obtain a compensator. Find the initial response of the compensator to the initial condition $\mathbf{x}(0) = [1; 0; 0; 0]^T$. Compare the required control inputs of the compensated system for the initial response to those required by the regulator in Exercise 8.24.

8.27. Using the MATLAB (CST) command *place*, design a full-order, two-output, current observer for the digitized aircraft lateral dynamics of Exercise 8.21 such that the observer poles are placed at $z_{1,2} = 0.8 \pm 0.2i$, $z_3 = 0.8$, and $z_4 = 0.1$. Combine the observer with the regulator designed in Exercise 8.25 to obtain a compensator. Find the initial response of the compensator to the initial condition $\mathbf{x}(0) = [0.5; 0; 0; 0]^T$. Compare the required inputs of the compensated system for the initial response to those required by the regulator in Exercise 8.25.

8.28. Repeat the compensator designs of Exercises 8.26 and 8.27 with *all* closed-loop poles at $z = 0$. Plot the deadbeat initial response of the designed closed-loop systems.

8.29. For the compensators designed in Exercise 8.28, simulate the response to measurement noise appearing simultaneously in all the output channels, using SIMULINK. (Use a *band-limited white noise* of power $10^{-6}$.)